THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, April 26, 1902, extending through the usual morning and afternoon sessions. Over forty persons were in attendance, including the following thirty-seven members of the Society:

Professor E. W. Brown, Dr. J. E. Clarke, Professor F. N. Cole, Mr. J. L. Coolidge, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Mr. Peter Field, Dr. William Findlay, Professor T. S. Fiske, Miss Carrie Hammerslough, Professor James Harkness, Professor L. I. Hewes, Dr. E. V. Huntington, Professor Harold Jacoby, Dr. S. A. Joffe, Dr. Edward Kasner, Dr. C. J. Keyser, Dr. G. H. Ling, Professor W. H. Metzler, Mr. H. B. Mitchell, Professor E. H. Moore, Dr. I. E. Rabinovitch, Professor J. K. Rees, Professor E. D. Roe, Miss I. M. Schottenfels, Professor I. J. Schwatt, Professor Charlotte A. Scott, Mr. Ferdinand Shack, Professor D. E. Smith, Dr. Virgil Snyder, Dr. H. F. Stecker, Dr. W. M. Strong, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor L. A. Wait, Miss E. C. Williams, Dr. Ruth G. Wood.

The President of the Society, Professor Eliakim Hastings Moore, occupied the chair, yielding it during the afternoon session to Professor T. S. Fiske. The Council announced the election of the following persons to membership in the Society: Professor C. E. Biklé, Columbia University; Professor F. W. Duke, Hollins Institute, Va.; Dr. J. G. Hardy, Williams College; Professor H. L. Hodgkins, Columbian University; Dr. J. N. Ivey, Tulane University; Dr. J. H. McDonald, University of California; Dr. H. C. Moreno, Stanford University; Dr. T. M. Putnam, University of California; Dr. E. W. Rettger, University of California; Mr. W. H. Roever, Harvard University; Professor Irving Stringham, University of California; Dr. S. D. Townley, University of California; Mr. H. E. Webb, Stevens School, Hoboken, N. J.; Mr. A. W. Whitney, University of California. Three applications for admission to membership were received.

The Council also authorized the organization of a Pacific Section of the Society, to hold meetings in the vicinity of San Francisco. The first meeting of the new Section was held May 3. A report of this meeting will appear in the next issue of the BULLETIN.

The following papers were read at the April meeting:

(1) Dr. H. F. STECKER: "The curve of least contour in the non-euclidean plane."

(2) Mr. J. L. COOLIDGE: "Quadric surfaces in hyperbolic space."

(3) Dr. F. H. SAFFORD: "Dupin's cyclides of the third degree."

(4) Mr. PETER FIELD: "On the forms of plane unicursal quintic curves."

(5) Dr. RUTH G. WOOD: "Non-euclidean displacements and symmetry transformations."

(6) Mr. D. R. CURTISS: "Note on the sufficient conditions for an analytic function."

(7) Miss I. M. SCHOTTENFELS: "On the definitional functional properties for the analytical functions

$$\frac{\sin \pi z}{\pi}, \frac{\cos \pi z}{\pi}, \frac{\tan \pi z}{\pi}."$$

(8) Professor C. A. SCOTT: "On the circuits of plane curves."

(9) Dr. E. V. HUNTINGTON: "A complete set of postulates for the theory of real numbers."

(10) Dr. L. P. EISENHART: "Surfaces whose lines of curvature in one system are represented on the sphere by great circles."

(11) Professor E. H. MOORE: "A definition of abstract groups."

(12) Dr. E. V. HUNTINGTON: "A second definition of a group."

(13) Mr. L. D. AMES: "Evaluation of slowly convergent series."

(14) Dr. EDWARD KASNER: "Groups of Cremona transformations and systems of forms."

(15) Mr. A. D. RISTEEN: "The constant of space."

(16) Dr. C. J. KEYSER: "Concerning the angles and the angular determination of planes in 4-space."

(17) Professor T. J. I'A. BROMWICH: "The infinitesimal generators of parameter groups."

(18) Professor T. J. I'A. BROMWICH: "On the parabolas, or paraboloids, through the points common to two given conics or quadrics."

Mr. Ames's paper was communicated to the Society through Professor Bôcher, Professor Bromwich's papers through the Secretary; Mr. Risteen was introduced by Professor P. F. Smith. In the absence of the authors, the

papers of Dr. Safford, Dr. Keyser, and Professor Bromwich were read by title and those of Mr. Curtiss and Mr. Ames were read by Dr. Huntington. The papers of Mr. Curtiss and Dr. Keyser appeared in the May number of the BULLETIN; those of Professor Bromwich and the second paper of Dr. Huntington are contained in the present number. Abstracts of the other papers are given below.

Dr. Stecker's paper applies the methods of the calculus of variations to the question of the curve of least contour in the non-euclidean plane, and establishes the fact that the curve sought is such that its non-euclidean distance from a fixed point is constant. It also follows that such a non-euclidean circle and the curve in the non-euclidean plane having constant Gaussian curvature are identical.

The metrical properties of a surface in hyperbolic space depend upon its relation to the absolute. A classification of the curves common to quadric surfaces and the absolute is given by Mr. Coolidge by means of Weierstrass's "elementary divisors." The various forms of surface are then taken up in detail and examined with regard to circular sections, focal conics, equations, and general appearance.

Dr. Safford's paper discusses cyclides derived from confocal cones of the second order and gives a method of generation by means of the diametral planes. The equations, in rectangular coördinates, of the three orthogonal families are considered, together with necessary conditions for the elimination of irrelevant conclusions as to intersections and lines of curvature.

The only papers devoted to the forms of plane unicursal quintic curves are two by Meyer and one by Dowling. In the first paper by Meyer, the method of knots is applied to curves having six real distinct crunodes, the forms of the curves are given, but no reference is made to equations or parametric representation. In Meyer's second paper the same results are obtained by quadric inversion. In the paper by Dowling one paragraph is devoted to the general equation of a unicursal quintic. No figures are given. Mr. Field completes the classification of unicursal quintics by a new method, giving the form and equation of each type. The definition of a type is essentially that proposed by Meyer. In the case of six distinct double points, a nodal cubic and a conic are used for basic curves. In case of a triple point, a line and a double conic have the same office; and finally

for a fourfold point the equation $u_4 + u_3 = 0$ is used. The positions of the real inflections are given for the last set.

The problem considered in Dr. Wood's paper has been suggested by Study in the *Leipziger Berichte* for 1890, and the results obtained for non-euclidean displacements and symmetry transformations are similar to those obtained by him for euclidean space. The equations of a non-euclidean line reflection (or skew reflection on conjugate polars of the quadric) are first found by means of the definition: A non-euclidean line reflection is the product of inversion in two complexes of the pencil $\alpha_i + \lambda\beta_i$, $\beta_i + \lambda\alpha_i$ $(i = 1, 2, 3)$ which are in involution. These equations are of the form

$$y_1' = \lambda(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \beta_1^2 - \beta_2^2 - \beta_3^2)y_1 + 2(\alpha_2\beta_3 - \alpha_3\beta_2)y_2 + 2(\alpha_3\beta_1 - \alpha_1\beta_3)y_3 + 2(\alpha_1\beta_2 - \alpha_2\beta_1)y_4,$$

and similarly for y_2', y_3', y_4', where $\sum_{i=1}^{3}\alpha_i\beta_i = 0$.

These equations can be found immediately from Cayley's formulæ for the homographic transformation of the quadric surface into itself if we set $\alpha_0 = \beta_0 = 0$. The law of composition of parameters of two such transformations given by Cayley furnishes an analytic proof of this theorem that every non-euclidean displacement (*i. e.*, a collineation of space which leaves the two systems of generators of the quadric surface invariant) may be compounded of two non-euclidean line reflections, and show that the directrices of the two line reflections belong to the complex which serves in defining the resulting displacement. The other two of Cayley's parameters α_0 and β_0 are found to be proportional to the non-euclidean distance and the cotangent of the non-euclidean distance between the directrices of the two line reflections of which the displacement is compounded. The equations for a non-euclidean symmetry transformation, *i. e.*, the collineations of space which interchange the two systems of generators of the quadric surface) are found by means of the theorem that such a transformation is compounded of a reflection upon an arbitrary plane as $x_1 = 0$ and a non-euclidean line reflection. These equations are the same as Cayley's "improper" transformation. The invariant planes, points, and the axis of rotation of such a transformation are readily found.

Miss Schottenfels's paper deals with the determination of the simplest set of definitional functional properties for the analytic functions

$$f(z) = \frac{\sin \pi z}{\pi}, \frac{\cos \pi z}{\pi}, \frac{\tan \pi z}{\pi}.$$

It is known that there exists a quartic with two nodes, having the property of being met by every straight line in at least two real points. The general theorem enunciated and proved in Professor Scott's paper is that for every order n there exist curves ($p = 0$ or 1), composed of a single circuit, met by every straight line in at least $n - 2$ real points; moreover, there exist curves composed of a single circuit, met by every straight line in at least $n - 4$, $n - 6$, $n - 8$, $\cdots$, real points. The proof is obtained by means of Cremona transformations. The nature of the curves is most clearly seen when they are projected on a sphere by lines through the center.

Dr. Huntington's paper on the postulates of all real numbers (positive, negative and zero) is a continuation of his paper on the postulates of magnitude (positive real number). published in the *Transactions* for April, 1902. In the present paper those assemblages are considered in which 1° a rule of combination $a \circ b$, and 2° a sub assemblage Q are so defined as to satisfy the following eight postulates:

1. If a and b are elements of the assemblage, there is an element x such that $a \circ x = b$.

2. If a, b, and either $a \circ b$ or $b \circ a$ belong to the assemblage, then $a \circ b = b \circ a$.

3. If a, b, c, $a \circ b$, $b \circ c$, and either $(a \circ b) \circ c$ or $a \circ (b \circ c)$ belong to the assemblage, then $(a \circ b) \circ c = a \circ (b \circ c)$.

4. If a and b belong to the sub-assemblage Q, then $a \circ b$, if it belongs to the assemblage at all, will also belong to Q.

5. If there is a peculiar element e such that $c \circ e = c$ for every element c, then e does not belong to Q.

6. If there is a peculiar element e such that $c \circ e = c$ for every element c, and if $a \circ b = e$ while $a \neq e$, then either a or b belongs to Q.

7. (Let the notation $a < b$ indicate that an element x of Q exists, such that $a \circ x = b$.) If S is an infinite sequence (a_k) of elements of Q, such that $a_k < a_{k+1}$ and $a_k < c$ ($k = 1, 2, 3, \cdots$), where c is some element of Q, then there is an element A having the following two properties: 1° $a_k < A$ whenever a_k belongs to S; 2° if $A' < A$ then there is at least one element of S, say a_r, for which $A' < a_r$.

8. If a is an element of Q, there are two elements of Q, say x and y, such that $x \circ y = a$.

Every assemblage which satisfies these eight postulates is

shown to be equivalent to the system of real numbers, and the independence of the postulates from one another is established.

Dr. Eisenhart's paper is in abstract as follows: Guichard has shown that the determination of all congruences, whose developables have a given spherical representation, reduces to the solution of an equation of Laplace, after the direction cosines of the lines have been found. We apply this method to the case where one family of lines on the sphere is composed of great circles and the other consists of their orthogonal trajectories. It is evident that all of these congruences are normal and that the lines on the sphere are the images of the lines of curvature of the parallel surfaces which cut the lines orthogonally. This furnishes a means for the determination and study of all surfaces whose lines of curvature in one system are represented on the sphere by great circles.

In the first place, it is found that, when such a configuration is given upon the sphere, the determination of the direction cosines of the lines of the congruence is the same problem as the finding of a skew curve from its intrinsic equations. For a given family of great circles there is a double infinity of families of surfaces whose lines of curvature in one system have this representation. Surfaces of this kind are characterized by the property that one family of lines of curvature are geodesics, and along the lines of curvature of the second system one of the principal radii is constant; and, furthermore, one of the sheets of the evolute is a developable surface and conversely. Hence these surfaces are the moulded surfaces of Dupin, and consequently can be generated by a plane curve whose plane rolls without sliding upon a developable surface. In the integration of the equation of Laplace two arbitrary functions are introduced; one of these determines the generating curve and the other joins with the spherical representation to fix the developable generator. The second sheet of the evolute is a moulded surface with the same surface generator and the generating curve is the evolute of the curve of the surface; this sheet and the involutes are represented on the sphere by the same great circles and corresponding points are at the distance of a quadrant.

Surfaces of revolution are moulded surfaces, corresponding to the case where the surface generator is a straight line. The complete determination of all the surfaces of revolution whose meridians are represented by a possible

family of great circles reduces to quadratures. Surfaces of revolution are the only moulded surfaces which are Weingarten surfaces and they are the only isothermic moulded surfaces.

The definition of abstract groups given by Professor Moore is an adaptation, in the light of that given by Dr. Huntington in the April (1902) BULLETIN, of a type of definition introduced independently some years ago by Professor Pierpont and himself. A set of elements $a, b, \cdots$, with a multiplication table, constitutes a group if: 1°. Every product $a \circ b$ belongs to the set. 2°. For every three elements a, b, c such that $a \circ b$, $b \circ c$, $a \circ (b \circ c)$, and $(a \circ b) \circ c$ belong to the set, $a \circ (b \circ c) = (a \circ b) \circ c$. 3_i°. An element i_l exists such that for every element a, $i_l \circ a = a$. 3_r°. An element i_r exists such that for every element a, $a \circ i_r = a$. 4_l°. If an element i_r (of character 3_r) exists, for some such element i_r for every element a an element a_l' exists such that $a_l' \circ a = i_r$. 5°. (a) The number of elements is a positive integer n, or (b) the set contains an infinitude of elements. The postulates 1, 2, 3_l, 3_r, 4_l, $5a$, and also the postulates 1, 2, 3_l, 3_r, 4_l, $5b$ are then shown to be independent of one another.

The following is an abstract of Mr. Ames's paper: Any convergent series

$$u_1 + u_2 + u_3 + \cdots \tag{1}$$

can be written

$$\frac{1}{2}u_1 + \frac{1}{2^2}(u_1 + u_2) + \frac{1}{2^3}(u_1 + 2u_2 + u_3) + \cdots, \tag{2}$$

or $\frac{1}{2}u_1 - \frac{1}{2^2}\Delta^1|u_1| + \frac{1}{2^3}\Delta^2|u_1| + \cdots$, where $\Delta^k|u_n|$ is the nth term of the kth order of differences of $|u_1| + |u_2| + \cdots$. This converges and equals (1). In many cases it converges rapidly when (1) converges slowly or diverges. Let R_k be the remainder in (2) after k terms. If (1) is alternating and $\Delta^k|u_n|$ and $\Delta^{k+1}|u_n|$ have opposite signs which remain unchanged for $n \geqq 1$, then $|R_k| < \frac{1}{2^k}\left|\Delta^k|u_n|\right|$. A sufficient condition for this is that $f(x)$ exists such that $f(n) = |u_n|$ and $\frac{d^k}{dx^k}f(x)$ and $\frac{d^{k+1}}{dx^{k+1}}f(x)$ have opposite signs which remain unchanged for $x > 1$. If the hypergeometric series $F(\alpha, \beta, \gamma, x)$ is convergent and $x < 0$, it always satifies the

first condition if a suitable number of terms be dropped at the beginning. This number can be determined in any particular case by computing the general term of the particular order of differences desired. The sum of the first K terms of (2) is $\frac{1}{2^k}[\lambda_1 u_1 + \lambda_2 u_2 + \cdots \lambda_n u_n]$ where $\lambda_r = 1 + k + \frac{k(k-1)}{1\ 2} + \cdots$ to $k - r + 1$ terms. In numerical computation it is usually better to compute the successive orders of differences. Simple inspection will in many cases make it highly probable without further tests that the degree of accuracy reached is much greater than that indicated above. Successive repetition of the process by which (2) was obtained from (1) gives

$$\frac{1}{p+1}u_0 + \frac{1}{(p+1)^2}(pu_0 + u_1) + \frac{1}{(p+1)^3}(p^2u_0 + 2pu_1 + u_2) + \cdots, \tag{3}$$

where p is arbitrary. Even when (1) is divergent, p can in many cases be so chosen that (3) converges. If (1) is a power series, and (3) converges throughout a larger region than (1), it yields the analytic continuation of the function represented by (1) throughout this larger region.

The groups considered in Dr. Kasner's paper are related to the various complete systems of invariants considered in the theory of forms. Let $f_1, f_2, \cdots, f_n$ be a set of forms of the same dimension and order, and $I_1, I_2, \cdots, I_r$ a system of invariants which is complete in the sense that any invariant of the forms is a rational function of the members of the system. Then if $f_{\alpha_1}, f_{\alpha_2}, \cdots, f_{\alpha_n}$ denote any permutation of the forms, and $I_1^{(\alpha)}, I_2^{(\alpha)}, \cdots, I_r^{(\alpha)}$ the corresponding system of invariants, this system is rationally expressible in terms of the original system, and the passage from the latter to the former constitutes a Cremona transformation. The transformations thus induced by all possible permutations are proved to form a group. The group is isomorphic to the symmetric permutation group of n letters, the isomorphism being in general simple, though in special cases it may be multiple.

All the groups obtained from the same set of forms by this process are equivalent within the infinite group consisting of all Cremona transformations. A distinct group is obtained, however, by means of the complete system of

absolute invariants. The binary linear forms give rise in this way to the cross ratio group discussed by Professor Moore in the *American Journal of Mathematics*, 1900. The groups discussed in detail are those relating to any set (n) of linear forms in any number (k) of variables for both the associated system and the system of absolute invariants, the number of variables being

$$\binom{n}{k} - \binom{n-k}{k} \quad \text{and} \quad (k-1)(n-k-1)$$

respectively. The process considered is capable of extension in several directions, for example to forms which are not all of the same order, and to the formation of infinite groups.

Mr. Risteen's paper is in abstract as follows: If Euclid's parallel axiom is true, then it is known that the parallax of a double star can be obtained in either of two ways, namely, 1° by the usual micrometric measures, and 2° by observing the relative velocities of approach or recession of the component stars in the line of sight. If the parallel axiom is not true, and space is admitted to be hyperbolic, then the known trigonometric relations that hold for hyperbolic space, and which were given by Lobachevsky, enable us to combine the micrometric measures and the spectroscopic ones, so as to obtain a single estimate of the star's distance, together with the value of the "constant of space" that occurs in Lobachevsky's equations, but whose value has not yet been determined. The same principle applies equally well if space is elliptic. Lobachevsky's method of finding a limiting value for his constant is unsound. A numerical value of the "constant of space" can hardly be found at present, because the necessary spectroscopic data cannot yet be had.

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THE INFINITESIMAL GENERATORS OF PARAMETER GROUPS.

BY PROFESSOR T. J. I'A. BROMWICH.

(Read before the American Mathematical Society, April 26, 1902.)

§1. Dr. Slocum has given (Bulletin, January, 1902, page 156) a method for calculating the infinitesimal gene-

rators of the parameter group which belongs to a group of known structure. On reading his paper it seemed to me that certain results of my own* might be applied with advantage; and on examination, I found that my process was less laborious than Slocum's and, I believe, more accurate. For I have found certain differences in our results, and, so far as I can test them, it seems that Slocum's results are less satisfactory than mine, in cases where the two differ.

§2. Slocum proves (l. c., page 158, equation (3)) that

$$\beta = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n!} \gamma_n$$

where

$$\gamma_n = (\alpha, \gamma_{n-1}), \qquad (n = 2, 3, 4, \cdots)$$

and

$$\gamma_1 = \gamma,$$

the parenthesis denoting Lie's alternant (or Klammerausdruck). Here α, β, γ are three differential operators of the given group; and Slocum shows that the determination of the differential operators of the parameter group is equivalent to finding the parameters of γ when those of β are given.

I shall now explain how to solve the last problem by applying certain theorems on matrices. To make the application clear, consider first the alternant (α, λ) where

$$\alpha = \sum_{k=1}^{r} a_k X_k, \quad \lambda = \sum_{k=1}^{r} l_k X_k$$

and the a's and l's are arbitrary parameters. If the structure of the given group is defined by

$$(X_p, X_q) = X_p X_q - X_q X_p = \sum_{s=1}^{r} c_{pqs} X_s \quad (p, q = 1, 2, \cdots, r),$$

we shall have

$$(\alpha, \lambda) = \Sigma_{p,q} a_p l_q (X_p, X_q) \quad (p, q = 1, 2, \cdots, r),$$

$$= \Sigma_{p,q} a_p l_q \left(\sum_{s=1}^{r} c_{pqs} X_s \right),$$

$$= \sum_{q=1}^{r} \sum_{s=1}^{r} a_{qs} l_q X_s,$$

where

* I should, perhaps, remark that these results were worked out in October, 1900, but were not published, owing to the advice of friends, who wished me to elaborate some further details. Unfortunately, I have not yet been able to carry this out In justice to Dr. Slocum, I ought to add that I had not then used my method for parameter groups.

$$a_{qs} = \sum_{p=1}^{r} a_p c_{pqs},$$

and so a_{qs} depends only on the a's and the structure of the given group. Hence the parameters of (a, λ) are derived from those of λ by means of a linear substitution; and this linear substitution has a matrix A, whose elements are a_{qs}.

In particular the parameters of γ_n are derived from those of γ_{n-1} by means of the substitution A; and hence (according to the definition of the square of a matrix) those of γ_n are derived from those of γ_{n-2} by means of A^2; and so on. Thus, finally, since $\gamma_1 = \gamma$, it follows that γ_n is derived from γ by means of a linear substitution whose matrix is A^{n-1}.

Turning now to the equation for β, it follows that the parameters of β are derived from those of γ by means of the substitution associated with the matrix

$$E - \frac{1}{2\,!}A + \frac{1}{3\,!}A^2 - \frac{1}{4\,!}A^3 + \cdots = \frac{[E - \text{exp.}\,(-A)]}{A}$$

$$= B, \text{ say,}$$

where E is the unit (or identical) matrix. Hence, to find the parameters of γ from those of β, we shall need the matrix B^{-1}, which is reciprocal to B. In terms of A, we have

$$B^{-1} = A/[E - \text{exp.}\,(-A)] = f(A) \text{ say.}$$

Now I come to the essential novelty of my method; in order to find B^{-1}, Slocum calculates B, and forms its reciprocal; but I shall show that *to find $B^{-1} = f(A)$, we need not find B first, but can find $f(A)$ directly from the properties of A and the form of the function $f(t)$, that is,*

$$f(t) = t/(1 - e^{-t}).$$

For, according to Frobenius and Stickelberger,* *any* function of a matrix (provided that such a function has a meaning) can be found as follows:

Take the matrix $(tE - A)$, where t is arbitrary, and form the reciprocal† matrix $(tE - A)^{-1}$; let the reciprocal be

* For references see § 2 of a paper by the present writer, *Proc. Camb. Phil. Soc.*, vol. 11 (1900), p. 75. The rule is there compared with the equivalent, but less convenient, rules found by Buchheim and Taber.

† If a_{qs} is an element of a matrix, the corresponding element of the reciprocal matrix is the minor of a_{sq} (with proper sign), divided by the determinant of the original matrix $|a_{sq}|$. In most of the cases considered here it is easier to construct the reciprocal by using the property that the "product" of a matrix by its reciprocal is E (*i. e.*, the unit matrix). It is perfectly easy to show that the two methods are equivalent.

arranged in a series of partial fractions, corresponding to the various factors of the determinant $|tE-A|$. Thus, suppose that, corresponding to a factor $(t-t_0)$, we get the terms in $(tE-A)^{-1}$

$$\frac{T_m}{(t-t_0)^m}+\frac{T_{m-1}}{(t-t_0)^{m-1}}+\cdots+\frac{T_1}{t-t_0},$$

where T_m, T_{m-1}, $\cdots$, T_1 are certain matrices (independent of t), and $(t-t_0)^m$ is an invariant factor of the determinant. Then we have, for any function

$$f(A)=\Sigma\left[f(t_0)T_1+f'(t_0)T_2+\cdots+\frac{1}{(m-1)!}f^{m-1}(t_0)T_m\right],$$

the summation being extended to all *distinct* factors of $|tE-A|$.

We shall apply this rule to find $B^{-1}=f(A)$.

One great advantage in this process is that we can separate the complications due to the form of $f(t)$ from those due to the form of A. Thus we should first calculate $f(t_0)$, $f'(t_0)$, $f''(t_0)$, $\cdots$, which will be available for *all* the groups; and the fresh calculations in each type relate only to the determination of T_1, T_2, T_3, $\cdots$. In practice, I find it less cumbrous to split up each element of the matrix $(tE-A)^{-1}$ into its partial fractions, rather than to write out the matrices T_1, T_2, T_3, $\cdots$ at length; then, wherever $1/(t-t_0)^{r+1}$ occurs, we have only to write $\frac{1}{r!}f^r(t_0)$ in its place, in order to deduce $f(A)$ from $(tE-A)^{-1}$.

For the special form of $f(t)$, which is required for the present purpose, we have, if $p=\frac{t_0}{1-e^{-t_0}}$,

$$f(t_0)=p,\quad f'(t_0)=p\left(1+\frac{1-p}{t_0}\right),$$

$$\frac{1}{2!}f''(t_0)=\frac{p^2(p-1)}{t_0^2}-\frac{3p^2}{2t_0}+\frac{p}{t_0}+\frac{p}{2}$$

and in the special case $t_0=0$, we have

$$f(0)=1,\quad f'(0)=\tfrac{1}{2},\quad \tfrac{1}{2}f''(0)=\tfrac{1}{12},\quad \tfrac{1}{6}f'''(0)=0,$$

where the continuation depends on Bernoulli's numbers. The coefficients given above are all that will be needed in the groups of *less* than five parameters. To include the case

of five parameters, we should want one more of each set; and so on.

Finally, when the matrix $f(A)$ is determined, it follows from Slocum's equation (7) that the kth operator of the parameter group has the coefficients found in the kth column of $f(A)$.

§ 3. To illustrate the method, I first took Type I of the three-parameter groups (page 161, l. c.). Here

$$(X_1, X_2) = X_1, \quad (X_1, X_3) = 2X_3, \quad (X_2, X_3) = X_3.$$

This gives for the matrix A

$$\begin{bmatrix} -a_2 & +a_1 & 0 \\ -2a_3 & 0 & +2a_1 \\ 0 & -a_3 & +a_2 \end{bmatrix}$$

and so

$$tE - A = \begin{bmatrix} t + a_2 & -a_1 & 0 \\ +2a_3 & t & -2a_1 \\ 0 & +a_3 & t - a_2 \end{bmatrix}.$$

Hence, forming the reciprocal, we find

$$(tE - A)^{-1} = \frac{1}{t(t^2 - \varphi^2)} \times \begin{bmatrix} t(t-a_2) + 2a_1a_3 & a_1(t-a_2) & 2a_1^2 \\ -2a_3(t-a_2) & t^2 - a_2^2 & 2a_1(t+a_2) \\ 2a_3^2 & -a_3(t+a_2) & t(t+a_2) + 2a_1a_3 \end{bmatrix}$$

where $\varphi^2 = a_2^2 - 4a_1a_3$, as used by Slocum; and it should be remarked that *every* element is divided by the denominator $t(t^2 - \varphi^2)$.

According to the rule we have now to divide each element into partial fractions, and then replace

$1/t$ by 1, $1/(t-\varphi)$ by $\varphi/(1-e^{-\phi})$, $1/(t+\varphi)$ by $\varphi/(e^{\phi}-1)$,

respectively, in order to obtain $f(A)$ from $(tE - A)^{-1}$.

It will save trouble, however, to put down the partial fractions for

$$\frac{1}{t(t^2-\varphi^2)}, \quad \frac{t}{t(t^2-\varphi^2)}, \quad \frac{t^2}{t(t^2-\varphi^2)}$$

separately. The reason for this is that every element of the matrix $(tE-A)^{-1}$ is a linear function of these three. Now

$$\frac{1}{t(t^2-\varphi^2)}=\frac{1}{\varphi^2}\left[-\frac{1}{t}+\frac{1}{2}\left(\frac{1}{t-\varphi}+\frac{1}{t+\varphi}\right)\right],$$

$$\frac{t}{t(t^2-\varphi^2)}=\frac{1}{2\varphi}\left(\frac{1}{t-\varphi}-\frac{1}{t+\varphi}\right),$$

$$\frac{t^2}{t(t^2-\varphi^2)}=\frac{1}{2}\left(\frac{1}{t-\varphi}+\frac{1}{t+\varphi}\right).$$

The first is therefore to be replaced by

$$-\frac{1}{\varphi^2}+\frac{1}{2\varphi^2}\left(\frac{\varphi}{1-e^{-\phi}}+\frac{\varphi}{e^{\phi}-1}\right)=\frac{1}{2\varphi^2}\frac{\varphi(e^{\phi}+1)-2}{e^{\phi}-1}=\tfrac{1}{2}\chi, \text{ say,}$$

the second is to be replaced by

$$\frac{1}{2\varphi}\left(\frac{\varphi}{1-e^{-\phi}}-\frac{\varphi}{e^{\phi}-1}\right)=\tfrac{1}{2}$$

and the third, by

$$\tfrac{1}{2}\left(\frac{\varphi}{1-e^{-\phi}}+\frac{\varphi}{e^{\phi}-1}\right)=\tfrac{1}{2}\varphi\frac{e^{\phi}+1}{e^{\phi}-1}=\tfrac{1}{2}\omega, \text{ say.}$$

I note that $\chi=\psi/\varphi^2$ in Slocum's notation. Thus

$$f(A)=\begin{bmatrix}\frac{1}{2}(\omega-a_3)+a_1a_2\chi & \frac{1}{2}(a_1-a_1a_3\chi) & a_1^2\chi\\ -a_2+a_2a_3\chi & \frac{1}{2}(\omega-a_3^2\chi) & a_1+a_1a_2\chi\\ a_2^2\chi & -\frac{1}{2}(a_3+a_3a_3\chi) & \frac{1}{2}(\omega+a_3)+a_1a_3\chi\end{bmatrix}.$$

Hence, by Slocum's theorem, the corresponding infinitesimal generators of the parameter group are found by reading *down* the columns of $f(A)$. That is, the generators are denoted by

$$A_1=[\tfrac{1}{2}(\omega-a_3)+a_1a_2\chi]\frac{\partial}{\partial a_1}+(-a_2+a_2a_3\chi)\frac{\partial}{\partial a_2}+a_2^2\chi\frac{\partial}{\partial a_3},$$

$$A_2=\tfrac{1}{2}(a_1-a_1a_3\chi)\frac{\partial}{\partial a_1}+\tfrac{1}{2}(\omega-a_3^2\chi)\frac{\partial}{\partial a_2}+\tfrac{1}{2}(-a_3-a_2a_3\chi)\frac{\partial}{\partial a_3},$$

$$A_3=a_1^2\chi\frac{\partial}{\partial a_1}+(a_1+a_1a_3\chi)\frac{\partial}{\partial a_2}+[\tfrac{1}{2}(\omega+a_3)+a_1a_3\chi]\frac{\partial}{\partial a_3},$$

which is Slocum's result, except that the coefficient of $\frac{\partial}{\partial a_2}$ in A_2 has a superficial difference, which disappears on using the equation $\varphi^2 = a_2^2 - 4a_1a_3$.

In order to exhibit my method in a more complicated case, I selected two of Slocum's results (type IV of the three-parameter groups; and type III of class B of the four-parameter groups). Here, however, our results did not agree; and I venture to give some details of my calculations, for the purpose of ready comparison.

Take first the three-parameter group with

$$(X_1, X_2) = 0, \quad (X_1, X_3) = X_1, \quad (X_2, X_3) = X_1 + X_2;$$

then I find the matrix

$$A = \begin{bmatrix} -a_3 & -a_3 & a_1 + a_2 \\ 0 & -a_3 & a_2 \\ 0 & 0 & 0 \end{bmatrix},$$

and so the matrix $(tE - A)^{-1}$ takes the form

$$\begin{bmatrix} \frac{1}{t+a_3} & \frac{-a_3}{(t+a_3)^2} & \frac{a_2}{(t+a_3)^2} + \frac{a_1}{a_3}\left(\frac{1}{t} - \frac{1}{t+a_3}\right) \\ 0 & \frac{1}{t+a_3} & \frac{a_2}{a_3}\left(\frac{1}{t} - \frac{1}{t+a_3}\right) \\ 0 & 0 & \frac{1}{t} \end{bmatrix}.$$

According to rule we are to put

$$1 \quad \text{in place of} \quad 1/t,$$

$$n = a_3/(e^{a_3} - 1) \quad \text{in place of} \quad 1/(t + a_3),$$

$$(n/a_3)(n + a_3 - 1) \quad \text{in place of} \quad 1/(t + a_3)^2,$$

and then the last matrix becomes $f(A)$.

It is perhaps unnecessary to write out $f(A)$ in full; but we deduce the generators

$$A_1 = n\frac{\partial}{\partial a_1}, \quad A_2 = -n(n + a_3 - 1)\frac{\partial}{\partial a_1} + n\frac{\partial}{\partial a_2}$$

$$A_3 = \left[\frac{a_1}{a_3}(1-n) + \frac{na_2}{a_3}(n+a_4-1)\right]\frac{\partial}{\partial a_1} + \frac{a_2}{a_3}(1-n)\frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3},$$

of which the third does not agree with Slocum's (the point of difference being the coefficient of $\frac{\partial}{\partial a_1}$). It may be remarked that these operators satisfy the relations,*

$$(A_1, A_2) = 0, \quad (A_1, A_3) = A_1, \quad (A_2, A_3) = A_1 + A_2,$$

and that the third of these is *not* satisfied by Slocum's operators. Indeed, it seems impossible for his operators to form a group; as (A_2, A_3) is not expressible in terms of A_1, A_2, A_3, when his symbols are used.

Next take the four-parameter group (type III on page 165, l. c.).

$$(X_1, X_2) = 0, \quad (X_1, X_3) = 0, \quad (X_2, X_3) = 0,$$
$$(X_1, X_4) = X_1, \quad (X_2, X_4) = X_1 + X_2, \quad (X_3, X_4) = X_2 + X_3.$$

Here I find the matrix $(tE - A)^{-1}$ in the form

$$\begin{bmatrix} \frac{1}{(t+a_4)} & \frac{-a_4}{(t+a_4)^2} & \frac{a_4^2}{(t+a_4)^3} & -\frac{a_3a_4}{(t+a_4)^3} + \frac{a_2}{(t+a_4)^2} + \frac{a_1}{a_4}\left(\frac{1}{t} - \frac{1}{t+a_4}\right) \\ 0 & \frac{1}{t+a_4} & \frac{-a_4}{(t+a_4)^2} & \frac{a_3}{(t+a_4)^2} + \frac{a_2}{a_4}\left(\frac{1}{t} - \frac{1}{t+a_4}\right) \\ 0 & 0 & \frac{1}{t+a_4} & \frac{a_3}{a_4}\left(\frac{1}{t} - \frac{1}{t+a_4}\right) \\ 0 & 0 & 0 & \frac{1}{t} \end{bmatrix}$$

Proceeding according to the rule, we now have the generators

$$A_1 = n\frac{\partial}{\partial a_1}, \quad A_2 = -n(n+a_4-1)\frac{\partial}{\partial a_1} + n\frac{\partial}{\partial a_2},$$

* Which are, as they ought to be, the same as the structural relations of the given group; this is a known property of parameter groups (Lie-Engel, Transformationsgruppen, vol. 1, p. 407).

$$A_3 = [n^2(n-1) + \tfrac{3}{2}n^2a_4 - na_4 + \tfrac{1}{2}na_4^2]\frac{\partial}{\partial a_1} - n(n + a_4 - 1)\frac{\partial}{\partial a_2} + n\frac{\partial}{\partial a_3},$$

$$A_4 = \Big[-n^2(n-1)\frac{a_2}{a_4} - \tfrac{3}{2}n^2a_3 + na_3 - \tfrac{1}{2}na_3a_4 + \frac{a_1}{a_4}(1-n) + \frac{a_2}{a_4}n(n + a_4 - 1)\Big]\frac{\partial}{\partial a_1} + \Big[\frac{a_3}{a_4}n(n + a_4 - 1) + \frac{a_2}{a_4}(1-n)\Big]\frac{\partial}{\partial a_2} + \frac{a_3}{a_4}(1-n)\frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

where $$n = a_4/(e^{a_4} - 1).$$

Here, comparing with Slocum's, the coefficients of $\frac{\partial}{\partial a_1}$ in A_2 and A_3 are of the opposite sign in the two calculations; while the coefficients of $\frac{\partial}{\partial a_1}$, $\frac{\partial}{\partial a_2}$ are almost entirely different in the two expressions for A_4.

I find that, with my results,

$$(A_1, A_2) = 0, \quad (A_1, A_3) = 0, \quad (A_2, A_3) = 0,$$

$$(A_1, A_4) = A_1, \quad (A_2, A_4) = A_1 + A_2, \quad (A_3, A_4) = A_2 + A_3,$$

which, of course, ought to be the case, on account of the general properties of the parameter group (*i. e.*, it has the same structural constants as the given group).

Taking Slocum's operators, I did not succeed in expressing (A_2, A_4) in terms of A_1, A_2, A_3, A_4; and it may be doubted if these operators can be generators of a group. But I did not calculate (A_3, A_4), for the evaluation of this alternant proved very laborious, when I used the operators found above, and it seemed superfluous to apply any further tests of this character.

§ 4. Having found these differences between my work and Slocum's, it seemed worth while to examine all the types; and I found the following divergences (in addition to those given before).

Class A (of the Four-Parameter Groups).

Type II.—The coefficient of $\frac{\partial}{\partial a_1}$ in A_3 comes out as

$$\frac{a_2[1-e^{a_4\beta}+\beta(1-e^{-a_4})e^{a_4\beta}]}{(e^{a_4\beta}-1)(e^{a_4(\beta-1)}-1)},$$

and that in A_4 is

$$\frac{a_3}{a_4}\left(1-\frac{a_4\beta}{e^{a_4\beta}-1}\right)+\frac{a_2a_3}{a_4(\beta-1)}\left[\frac{1}{e^{a_4}-1}+\frac{\beta(\beta-2)}{e^{a_4\beta}-1}-\frac{(\beta-1)^2}{e^{a_4(\beta-1)}-1}\right].$$

Type III.—The first term in A_3 appears as*

$$\left\{\frac{2a_2}{(e^{a_4}-1)^2}[(a_4-1)e^{a_4}+1]+\frac{a_3-2a_2}{e^{a_4}+1}\right\}\frac{\partial}{\partial a_1},$$

while the first and second in A_4 are

$$\frac{1}{a_4}\left\{a_1\left(1+\frac{a_4}{e^{a_4}-1}\right)+2a_2^2\left[\frac{(a_4-1)e^{a_4}+1}{(e^{a_4}-1)^2}-\frac{1}{e^{a_4}+1}\right]\right\}\frac{\partial}{\partial a_1}$$
$$+\left[\frac{a_2}{a_4}\left(1-\frac{a_4}{e^{a_4}-1}\right)+2a_2\frac{(a_4-1)e^{a_4}+1}{(e^{a_4}-1)^2}\right]\frac{\partial}{\partial a_2}.$$

Type IV.—The coefficient of $\frac{\partial}{\partial a_1}$ in A_3 comes out as

$$-\frac{a_2[e^{a_4}(a_4-1)+1]}{(e^{a_4}-1)^2}=-\frac{na_2}{a_4}(n+a_4-1),$$

and that in A_4 as

$$\frac{a_3}{a_4}(1-n)+\frac{a_2a_3}{a_4^2}(n^2+a_4n-1),$$

where

$$n=a_4/(e^{a_4}-1).$$

Class B (of the Four-Parameter Groups).

Type II.—The coefficient of $\frac{\partial}{\partial a_2}$ in A_3 appears to be equal, but of the opposite sign, to Slocum's; while that in A_4 is

$$\frac{a_2}{a_4}(1-n)+\frac{na_3}{a_4\beta}(n+a_4\beta-1),$$

where

$$n=a_4\beta/(e^{a_4\beta}-1).$$

Type III.—See § 3, above.

Type V.—The coefficient of $\frac{\partial}{\partial a_2}$ in A_4 comes out as

* Although I agree with Slocum's expression for A_3 in this type, yet it seems neater to put the coefficient of $\frac{\partial}{\partial a_1}$ in the shape $\left(-\frac{a_3}{e^{a_4}+1}\right)$.

$$\frac{a_3}{a_4}\left(1-\frac{a_4}{e^{a_4}-1}\right)+\frac{a_3[(a_4-1)\,e^{a_4}+1]}{(e^{a_4}-1)^2}.$$

Owing to the large amount of arithmetic involved in the deduction of these results, I can hardly hope to have carried them through without error. Unfortunately, I have not been able to obtain any assistance in verifying the work; and all my calculations have had to be made under the pressure of professorial class-work. However, in all cases where differences showed themselves between my first results and those of Slocum, my work has been carried out twice (and, in some cases, three times); and all checks have been applied that suggested themselves (see, for instance, § 5 below).

§ 5. It may be of use to remark that if the matrix A belonging to one group can be obtained from that of a second group by specializing a certain constant, then it is possible to obtain the matrix $f(A)$ of the first group, from that of the second, by using the same specialized constant, *provided that this change does not affect the invariant factors of* $|tE-A|$. As the parameter operators depend only on $f(A)$, it follows that the parameter operators of the first given group must be deducible similarly from that of the second group.

Thus, taking the three-parameter groups, type III has a matrix A which is deduced from that of type II by writing $\beta=1$; and this does not modify the invariant factors of $|tE-A|$; hence the parameter operators of III follow from those of II by putting $\beta=1$. Similarly V comes from II, by putting $\beta=0$.

So also, in Class A of the four-parameter groups, type IV comes from II, by writing $\beta=1$; that is, if the alterations mentioned in § 4 are made first; Slocum's results are inconsistent here. Further, in class B, types IV and VI come from I, by putting (i) $\beta=\alpha$, and (ii) $\gamma=\beta=\alpha=1$. Also, type V is obtained from II by writing $\beta=\alpha=1$; but here Slocum's operators must be modified (see § 4) before this can be verified.

§ 6. Concluding, it may be added that it is also possible to find $f(A)$ directly from the series in which we can expand the function

$$A/[E-\exp(-A)]=E+\tfrac{1}{2}A+\tfrac{1}{12}A^2+\cdots.$$

To do so, we must make use of the characteristic equa-

tion * of the matrix A, in order to express the higher powers of A. This leads to tedious work unless the characteristic equation reduces to one in two terms only; for instance, types V and VI (in Slocum's list of three-parameter groups) lead to the equations for A

$$A^2 + a_3 A = 0, \quad A^2 = 0$$

respectively. This method has been used by Dr. H. F. Baker in a recent paper on the calculation of the finite equations of a group from its structural constants (*Proceedings of the London Mathematical Society*, volume 34 (1902), page 91); but Baker's work relates solely to the determination of the matrix which is denoted by B in my notation, *i. e.*, the matrix reciprocal to $f(A)$.

I have not actually applied the last method to any of the harder cases; indeed, I have only used it for the two cases just mentioned, when it gives

Type V. $$f(A) = E + \left(\frac{1}{a_3} - \frac{1}{e^{a_3} - 1}\right) A.$$

Type VI. $$f(A) = E + \tfrac{1}{2} A.$$

It may, however, prove useful as an alternative means of verification.

QUEEN'S COLLEGE, GALWAY,
February 25, 1902.

ON THE PARABOLAS (OR PARABOLOIDS) THROUGH THE POINTS COMMON TO TWO GIVEN CONICS (OR QUADRICS).

BY PROFESSOR T. J. I'A. BROMWICH.

(Read before the American Mathematical Society, April 26, 1902.)

IN the December issue of the BULLETIN (page 122, December, 1901) Huntington and Whittemore have called attention to the features of conics which touch the line infinity

* This is obtained as follows: Let $\phi(t)$ be the quotient of the determinant $|tE - A|$ by the highest common factor of all its first minors. Then the equation is

$$\phi(A) = 0.$$

See Frobenius, *Crelle*, vol. 84; and other references given by the present author in a review, BULLETIN, vol. 7 (1900), p. 308.

at one of the circular points. Apparently they were led to the discussion of these conics by considering the disappearance of parabolas from the family $S + \lambda S' = 0$, when $S = 0$, $S' = 0$ are rectangular hyperbolas. This recalls to my mind a neat result which I have used in class-work, but have never seen in print, namely:

The two parabolas contained in the family $S + \lambda S' = 0$ are real, if either of the two conics $S = 0$, $S' = 0$ is an ellipse; this condition is sufficient but not necessary.

This is at once obvious if we write in cartesians

$$S = ax^2 + 2hxy + by^2 + \cdots, \quad S' = a'x^2 + 2h'xy + b'y^2 + \cdots,$$

for then the condition for a parabola is

$$\begin{vmatrix} a - \lambda a', & h - \lambda h' \\ h - \lambda h', & b - \lambda b' \end{vmatrix} = 0,$$

which will have real roots if either $ab - h^2 > 0$, or $a'b' - h'^2 > 0$ (by Weierstrass's general theorem, *Monatsberichte d. k. Akad. z. Berlin*, 1858; Werke, volume 1, page 233, §3).

Suppose now that S and S' are hyperbolas; then a short piece of arithmetic will prove that the *necessary* condition that the values of λ may be real is that the two pairs of lines

$$ax^2 + 2hxy + by^2 = 0, \quad a'x^2 + 2h'xy + b'y^2 = 0$$

do *not* interlace (cf. Chrystal's *Algebra*, volume 1, p. 464). These pairs of lines are, of course, parallel to the asymptotes of S and S', and so, if S, S' are rectangular hyperbolas, it is clear that the parabolas of the system must be imaginary.

Proceeding next to the possibility of *coincident* parabolas in the family $S + \lambda S'$, we have first, when S or S' is an ellipse, that the equation for λ has equal roots only if

$$\frac{a}{a'} = \frac{h}{h'} = \frac{b}{b'} = \lambda,$$

as is seen by a second theorem due to Weierstrass (*l. c.*, §4). In this case, each parabola degenerates to a line and the line at infinity.

Second, if S, S' are *both* hyperbolas, we may have the same result as just given. Or, we may have one asymptote of S parallel to one of S', and then the coincident parabolas of $S + \lambda S'$ are not necessarily degenerate; the axis of the

resulting parabola is parallel to the two parallel asymptotes.

An extension to space of three dimensions is easy; thus the analogue to the first theorem gives

The three paraboloids contained in the family $S + \lambda S' = 0$ are all real, if either of the quadrics S, S' is an ellipsoid. So too, we find:

If the two quadrics S, S' are hyperboloids, two of the paraboloids will be imaginary if (and only if) two cones with a common vertex, parallel to their asymptotic cones, intersect in *two* real generators.

There are five possibilities when two (or three) of the paraboloids coincide; without enumerating them all, it may be noted that when S or S' is an ellipsoid, the coincidence implies degeneration of the paraboloids. All the other cases may be obtained by suitable interpretations of Weierstrass's algebra ("Zur Theorie der bilinearen und quadratischen Formen," *Monatsberichte d. k. Akad. z. Berlin*, 1868; Werke, volume 2, page 19).

Slightly digressing from the line of thought just indicated, and reverting to Huntington and Whittemore's paper, I note that their result, that the eccentricity is wholly indeterminate (*l. c.*, page 123), *suffices* to specify the conics considered by them. For, in orthogonal cartesians, the eccentricity is determined by the ratio $(a+b)^2/(ab-h^2)$, which is only indeterminate if

$$ab - h^2 = 0, \quad a + b = 0,$$

i. e., if

$$b = -a, \quad h = \pm ia,$$

and then the conic reduces to

$$a(x \pm iy)^2 + \text{linear terms} = 0.$$

QUEEN'S COLLEGE, GALWAY,
February 22, 1902.

A SECOND DEFINITION OF A GROUP.

BY DR. E. V. HUNTINGTON.

(Read before the American Mathematical Society, April 26, 1902.)

THE following note contains a definition of a group expressed in four independent postulates, suggested by the definition given in W. Burnside's Theory of Groups of Finite Order (1897). The definition presented by the writer at the February meeting contained three independent

postulates,* and the definition just proposed by Professor Moore † contains five independent postulates. The comparison of these three definitions is therefore very striking.

Definition.

We consider here an assemblage or set of elements in which a rule of combination* denoted by $\circ$ is so defined as to satisfy the following four postulates:

1. *If a and b belong to the assemblage, then $a \circ b$ also belongs to the assemblage.*

2. $(a \circ b) \circ c = a \circ (b \circ c)$, *whenever $a \circ b$, $b \circ c$, $(a \circ b) \circ c$ and $a \circ (b \circ c)$ belong to the assemblage.*

3. *For every two elements a and b there is an element a' such that $(a \circ a') \circ b = b$.*

4. *For every two elements a and b there is an element a'' such that $b \circ (a'' \circ a) = b$.*

From 1, 2, 3 it follows that for any two elements a and b there is an element x such that $a \circ x = b$. For by 3 take a' so that $(a \circ a') \circ b = b$ and by 1 take $x = a' \circ b$; then by 2 $a \circ x = b$.

Similarly, from 1, 2, 4 follows the existence, for every two elements a and b, of an element y such that $y \circ a = b$.

Therefore every assemblage which satisfies the postulates 1, 2, 3, 4 is a group, according to the writer's previous definition.

If we wish to distinguish between finite and infinite groups we may add a fifth postulate, either:

5a. *The assemblage contains n elements, where n is a positiv integer;*‡ or

5b. *The assemblage contains an infinitude of elements.*

Independence of Postulates 1, 2, 3, 4 and 5a, when $n > 2$.

The mutual independence of postulates 1, 2, 3, 4, 5a for finite groups may be established, when $n > 2$, by use of the following systems:

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*See BULLETIN, pp. 296–300.

† An abstract of Professor Moore's paper is given on p. 373 of the present number of the BULLETIN.

‡ The number of elements in a finite group is called the *degree* of the group by H. Weber, Algebra, Vol. II (1899), p. 4, or the *order* of the group by most other writers. Cf. W. Burnside, *loc. cit.*, p. 380.

M_1. If n is odd, $n = 2k + 1$, let M_1 be the system of al integers from $-k$ to $+k$, while $a \circ b = a + b$.

If n is even, $n = 2k + 2$, let M_1 be the system of all integers from $-k$ to $+k$ with an additional element z, while the rule of combination is defined as follows: When $a \neq z$ and $b \neq z$, $a \circ b = a + b$; further, $z \circ 0 = 0 \circ z = z$, and $z \circ z = 0$; but when $a \neq 0$, $a \circ z = z \circ a = k + 1$ which does not belong to the assemblage.

M_2. Let M_2 be the system of positive integers from 1 to n, with the rule of combination defined as follows:

$$\begin{aligned} a \circ b &= a + b && \text{when } a + b \leqq n, \\ &= a + b - n && \text{when } a + b > n, \end{aligned}$$

except that $a \circ b = 2$ when $a + b = 1$ or $n + 1$,
and $a \circ b = 1$ when $a + b = 2$ or $n + 2$.

M_3. The system of positive integers from 1 to n, with $a \circ b = a$.

M_4. The system of positive integers from 1 to n, with $a \circ b = b$.

M_5. Any infinite group.

Since the system M_k is found to satisfy all the other postulates but not the kth ($k = 1, 2, 3, 4, 5$) we see that no one of these five postulates is a consequence of the remaining four.

Independence of Postulates 1, 2, 3, 4 and 5b.

Similarly, the mutual independence of postulates 1, 2, 3, 4, 5*b* for infinite groups may be established by the use of the following systems:

N_1. The system of all integers except $+1$ and -1, with $a \circ b = a + b$.

N_2. The system of all rational numbers, with $a \circ b = (a + b)/2$.

N_3. The system of all positive integers, with $a \circ b = a$.

N_4. The system of all positive integers, with $a \circ b = b$.

N_5. Any finite group.

Thus no one of these five postulates is a consequence of the remaining four.

Weber's Definition of a Finite Group.

In conclusion we may notice that if, in the definition of a *finite* group, we replace postulates 3 and 4 by the following:

3′. *If* $a \circ b = a \circ b'$ *then* $b = b'$;

4′. *If* $a \circ b = a' \circ b$ *then* $a = a'$;

we shall have the definition given by H. Weber, *loc. cit.* That these postulates 1, 2, 3′, 4′, 5*a* are mutually independent (when $n > 2$) has already been shown in the writer's previous paper (page 300).

It should be noticed, however, that postulates 1, 2, 3′, 4′, 5*b* would not be sufficient to define an *infinite* group, since the system of positive integers, with $a \circ b = a + b$, satisfies them all, and is not a group.

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DETERMINATION OF ALL THE GROUPS OF ORDER p^m, p BEING ANY PRIME, WHICH CONTAIN THE ABELIAN GROUP OF ORDER p^{m-1} AND OF TYPE $(1, 1, 1, \cdots)$.

BY PROFESSOR G. A. MILLER.

(Read before the San Francisco Section of the American Mathematical Society, May 3, 1902.)

LET $t_1, t_2, \cdots, t_{m-1}$ represent a set of independent generators of the abelian group H of type $(1, 1, 1, \cdots)$. It is well known that the order of the group of isomorphisms ϑ of H is $p^{\frac{(m-1)(m-2)}{2}}(p-1)(p^2-1)\cdots(p^{m-1}-1)$. One of its subgroups ϑ_1 of order $p^{\frac{(m-1)(m-2)}{2}}$ is composed of all the operators of ϑ which correspond to the holomorphisms of H in which t_a $(a = 2, 3, \cdots, m-1)$ corresponds to itself multiplied by some operator in the group generated by $t_1, t_2, \cdots, t_{a-1}$. The number of conjugates of ϑ_1 under ϑ is clearly equal to the order of ϑ divided by $p^{\frac{(m-1)(m-2)}{2}}(p-1)^{m-1}$. We shall first determine the number of sets of subgroups of ϑ_1 which are conjugate under ϑ. It may be observed that even characteristic subgroups of ϑ_1 may be conjugate under ϑ. For instance, the octic group has a characteristic subgroup of order two and four other subgroups of this order, yet all of these subgroups are conjugate under ϑ when the latter is the simple group of order 168.

All the holomorphisms of H may be obtained by establishing isomorphisms between H and its subgroups and letting the product of two corresponding operators in these isomorphisms correspond to the original operator of H.*

* BULLETIN, vol. 6 (1900), p. 337.

If two operators of ϑ_1 correspond to such holomorphisms in which the subgroups are of different orders, they evidently belong to different sets of conjugates under ϑ. Hence $m-2$ sets of conjugate subgroups of ϑ_1 can be obtained in this manner. It remains to determine the number of sets of conjugates when the multiplying subgroups are of the same order h_1. When h_1 is either p or p^{m-2} it is not difficult to see that the corresponding operators of ϑ_1 are conjugate under ϑ.

In general, suppose that in any holomorphism of H, which corresponds to an operator in ϑ whose order is a power of p, the above set of independent generators of H has been so selected that those which correspond to themselves come first; and let t_a' represent any operator of the group generated by $t_1, t_2, \cdots, t_a$ which is not in its subgroup generated by $t_1, t_2, \cdots, t_{a-1}$. The operator of ϑ which corresponds to this holomorphism transforms some t_a', which is not found in the multiplying subgroup, into $t_{a_1}t_a'$, t_{a_1} into $t_{a_2}t_{a_1}$, $\cdots$, $t_{a_{n-1}}$ into $t_{a_n}t_{a_{n-1}}$, and t_{a_n} into itself. If $t_{a_1}, t_{a_2}, \cdots, t_{a_n}$ do not generate the multiplying subgroup, *i. e.*, if h_1 exceeds p^n, we may find another series of transforms in the same manner and thus arrive at another number n_1. The necessary and sufficient condition that two holomorphisms may correspond to conjugate operators under ϑ is that the numbers $n, n_1, \cdots$ and $n', n_1', \cdots$ which correspond to these holomorphisms are the same.

For instance, when $m=5$, ϑ_1 contains four sets of conjugate cyclic subgroups under ϑ. Two of these correspond to the case where the multiplying subgroup is of order p^2 and the other two correspond to the cases where this order is p or p^3. When $m=6$, the number of these conjugate sets is clearly six. In general, *when $h_1 = p^\lambda$ the number of sets of cyclic subgroups of ϑ_1, such that each set includes one complete system of conjugates under ϑ, is equal to the number of partitions of λ with respect to addition when the number of addends does not exceed $m-1-\lambda$.* Hence, it follows from Euler's theorem: the number of partitions of $n+r$ into r parts is equal to the number of ways in which n can be represented as the sum of one or more of the r numbers $1, 2, \cdots, r$ (with repetitions), that the total number of such sets of cyclic subgroups in ϑ_1 is equal to the number of partitions of $m-1$ into at least two parts with respect to addition. This number does not include the identity.

The orders of the operators of ϑ_1 can be directly obtained by means of the formula*

* BULLETIN, vol. 7 (1901), p. 351.

$$t^{-\alpha}s_\alpha t^\alpha = s_{\alpha+\alpha}s^\alpha_{\alpha+\alpha-1}\cdots s^{\alpha(\alpha-1)\cdots(\alpha-r+1)/r!}_{\alpha+\alpha-r}\cdots s^\alpha_{\alpha+1}s_\alpha$$

whenever $t^{-1}s_\beta t = s_{\beta+1}s_\beta$ $(\beta = a, a+1, \cdots, a+n-1)$. If i_1 represents any operator of θ_1 which corresponds to the numbers $n, n_1, \cdots$ its order is p^{α_1}, where α_1 is the smallest power of p which exceeds each of the numbers $n, n_1, \cdots$. When $\alpha_1 > 1$ the conjugate set to which i_1 belongs does not give rise to any group of order p^m. When $n = 1$ and all the other numbers $n_1, \cdots$ are zero, the conjugate set to which i_1 belongs gives rise to three groups whenever $p > 2$ and $m > 3$.* One of these is conformal with the abelian group of type $(1, 1, 1, \cdots)$ while the other two are conformal with the abelian group of type $(2, 1, 1, \cdots)$. In one of the last two groups each of the p^{m-2} cyclic subgroups of order p^2 is invariant while none of these subgroups is invariant in the other.

When p is odd and $m = 3$ there are just two groups, which are conformal respectively with the abelian groups of types $(1, 1, 1, \cdots)$ and $(2, 1, 1, \cdots)$. Each of the p^{m-2} cyclic subgroups of order p^2 in the latter group is invariant. When $p = 2$ and $m = 3$ there is only one group, viz., the octic group; when $m > 3$ there are two groups, one being conformal with the abelian group of type $(2, 1, 1, \cdots)$ while the other contains $2^{m-1} + 2^{m-2} - 1$ operators of order 2, the remaining operators being of order 4. Each of the groups which have been considered contains $p + 1$ abelian subgroups of order p^{m-1} while remaining groups contain no abelian subgroups of this order except H.

In general, if p exceeds all the numbers $n + 1, n_1 + 1, \cdots$, then i_1 and H generate a group which contains only operators of order p besides identity. For, if s is any operator of H, then

$$(st)^p = st\,st\,st\cdots = st\,st^{-1}t^2\,st^{-2}t^3\cdots t^{p-1}st = 1.$$

For each of the numbers $n, n_1, \cdots$ which are distinct there is clearly one additional group in which H is transformed in the same manner as i_1 transforms it. If the sum of the numbers $n + 1, n_1 + 1, \cdots$ is less than $m - 1$; *i. e.*, if H contains invariant operators which are not commutators, there is one additional group in which H is transformed according to i_1. All of these groups are conformal with the abelian group of type $(2, 1, 1, 1, \cdots)$.

When p is equal to one or more of the numbers $n + 1$,

*Only non-abelian groups of order p^m are considered in this paper, since the two abelian groups which contain H are well known.

$n_1 + 1, \cdots$. then the group generated by i_1 and H contains operators of order p^2 and the remarks in regard to additional groups apply only to the remaining numbers and to the invariant operators of H which are not commutators. As i_1 and its conjugates cannot give rise to any group of order p^m when p is less than some one of the numbers $n+1$, $n_1 + 1, \cdots$, all the groups of this order which contain H can be readily obtained by the above considerations. It may be observed that this includes all the groups of order p^m in which every operator is of order p whenever $m < 5$, since every group of order p^4 contains an abelian subgroup of order p^3.

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A CLASS OF SIMPLY TRANSITIVE LINEAR GROUPS.

BY PROFESSOR L. E. DICKSON.

1. IN the study of the group defined for any given field by the multiplication table of any given finite group,* it is necessary to discuss the types of simply transitive linear homogeneous groups G whose transformations can be given the form

$$\begin{aligned} &\xi_1' = \eta_1\xi_1, \quad \xi_2' = \eta_2\xi_1 + \eta_1\xi_2, \quad \xi_3' = \eta_3\xi_1 + a\xi_2 + \eta_1\xi_3, \\ (1) \qquad &\xi_4' = \eta_4\xi_1 + \beta\xi_2 + \gamma\xi_3 + \eta_1\xi_4, \\ &\xi_5' = \eta_5\xi_1 + \lambda\xi_2 + \mu\xi_3 + \nu\xi_4 + \eta_1\xi_5, \cdots. \end{aligned}$$

Here $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \cdots$ are the independent parameters, while $a, \beta, \gamma, \lambda, \cdots$ are linear homogeneous functions of the η_i. Burnside† was led to the erroneous conclusion that every such group G is an abelian group. He first concludes that the expression for ξ_i' contains only the parameters $\eta_1, \cdots, \eta_i$ and contains η_i only in the first term $\eta_i\xi_1$. That this result need not be true is shown by a consideration of the simply transitive group of quaternary transformations

$$\begin{aligned} &\xi_1' = \eta_1\xi_1, \quad \xi_2' = \eta_2\xi_1 + \eta_1\xi_2, \quad \xi_3' = \eta_3\xi_1 + a\xi_2 + \eta_1\xi_3, \\ (2) \qquad &\xi_4' = \eta_4\xi_1 - \frac{a_3}{a_4}a\xi_2 + \eta_1\xi_4, \end{aligned}$$

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* For the case of a continuous field, Burnside, *Proc. Lond. Math. Soc.*, vol. 29 (1898), pp. 207–224, 546–565; for an arbitrary field, Dickson, *Transactions*, vol. 3 (1902), pp. 285–301.

† *Proc. Lond. Math. Soc.*, vol. 29, pp. 552–553.

where $a \equiv a_2\eta_2 + a_3\eta_3 + a_4\eta_4$, $a_4 \neq 0$. Let Y_i be the infinitesimal transformation obtained by setting $\eta_i = \delta t$, $\eta_j = 0$ $(j = 1, 2, 3, 4\,;\ j \neq i)$. Then

$$Y_1 = \sum_{i=1}^{4} \xi_i \frac{\partial f}{\partial \xi_1}, \quad Y_2 = \xi_1 \frac{\partial f}{\partial \xi_2} + a_2\xi_2 \frac{\partial f}{\partial \xi_3} - \frac{a_3}{a_4} a_2 \xi_2 \frac{\partial f}{\partial \xi_4},$$

$$Y_3 = (\xi_1 + a_3\xi_2) \frac{\partial f}{\partial \xi_3} - \frac{a_3^2}{a_4} \xi_2 \frac{\partial f}{\partial \xi_4},$$

$$Y_4 = a_4\xi_2 \frac{\partial f}{\partial \xi_3} + (\xi_1 - a_3\xi_2) \frac{\partial f}{\partial \xi_4}, \quad (Y_2Y_3) = a_2 Y_2 - \frac{a_3^2}{a_4} Y_4,$$

$$(Y_2Y_4) = a_4 Y_3 - a_3 Y_4, \quad (Y_3Y_4) = 0, \quad (Y_1Y_i) = 0.$$

But the desired result can always be reached by applying a suitable transformation on the variables ξ_i and the cogredient transformation on the parameters η_i (see §§ 4, 5 below). Taking G in this reduced form, Burnside attempts to prove by induction that G is abelian. He supposes that the first $t-1$ equations of G define an abelian group and concludes, from the fact that G is its own parameter group, that the subgroup of G generated by the infinitesimal operations Y_1, Y_2, ..., Y_{t-1} corresponding to η_1, η_2, ..., η_{t-1} is abelian. The invalidity of the conclusion is shown by an example. The transformations

$$(3) \qquad \begin{aligned} &\xi_1' = \eta_1\xi_1, \quad \xi_2' = \eta_2\xi_1 + \eta_1\xi_2, \quad \xi_3' = \eta_3\xi_1 + \eta_1\xi_3, \\ &\xi_4' = \eta_4\xi_1 + (b_2\eta_2 + b_3\eta_3)\,\xi_2 + (c_2\eta_2 + c_3\eta_3)\xi_3 + \eta_1\xi_4 \end{aligned}$$

constitute a simply transitive group in reduced form, which is its own parameter group. The first three equations taken alone constitute an abelian group. But Y_1, Y_2, Y_3 do not generate a group if $b_3 \neq c_2$. In fact,

$$Y_1 = \sum_{i=1}^{4} \xi_i \frac{\partial f}{\partial \xi_i}, \quad Y_2 = \xi_1 \frac{\partial f}{\partial \xi_2} + (b_2\xi_2 + c_2\xi_3) \frac{\partial f}{\partial \xi_4},$$

$$Y_3 = \xi_1 \frac{\partial f}{\partial \xi_3} + (b_3\xi_2 + c_3\xi_3) \frac{\partial f}{\partial \xi_4}, \quad Y_4 = \xi_1 \frac{\partial f}{\partial \xi_4},$$

$$(Y_1Y_2) = 0, \quad (Y_1Y_3) = 0, \quad (Y_2Y_3) = (b_3 - c_2)\,Y_4,$$

$$(Y_iY_4) = 0 \qquad (i = 1, 2, 3).$$

Each of the two preceding examples shows that G need not be abelian.

2. For the case of one variable or the case of two variables,

the transformations (1) evidently form a simply transitive abelian group. We proceed to consider the cases $n = 3, 4, 5$. The method applies immediately to n variables, but the formulæ are complicated by the necessary use of a triple subscript notation for the coefficients. Set

$$\alpha = \sum_{k=1}^{n} a_k \eta_k, \quad \beta = \Sigma b_k \eta_k, \quad \gamma = \Sigma c_k \eta_k,$$

$$\lambda = \Sigma l_k \eta_k, \quad \mu = \Sigma m_k \eta_k, \quad \nu = \Sigma n_k \eta_k,$$

where the $a_k, \cdots, n_k$ are constants for a particular group G. The general transformation of G will be designated T_η or $T_{\eta_1, \cdots, \eta_n}$. Then the product $T_{\eta'} T_\eta$ is of the form (1) with η_i'', α'', β'', $\cdots$, in place of η_i, α, β, where

$$\eta_1'' = \eta_1\eta_1', \quad \eta_2'' = \eta_2\eta_1' + \eta_1\eta_2', \quad \eta_3'' = \eta_3\eta_1' + \alpha\eta_2' + \eta_1\eta_3',$$

$$\eta_4'' = \eta_4\eta_1' + \beta\eta_2' + \gamma\eta_3' + \eta_1\eta_4',$$

$$\eta_5'' = \eta_5\eta_1' + \lambda\eta_2' + \mu\eta_3' + \nu\eta_4' + \eta_1\eta_5',$$

$$\alpha'' = \alpha\eta_1' + \eta_1\alpha', \quad \beta'' = \beta\eta_1' + \gamma\alpha' + \eta_2\beta', \quad \gamma'' = \gamma\eta_1' + \gamma'\eta_1,$$

$$\lambda'' = \lambda\eta_1' + \mu\alpha' + \nu\beta' + \eta_1\lambda', \quad \mu'' = \mu\eta_1' + \nu\gamma' + \eta_3\mu',$$

$$\nu'' = \nu\eta_1' + \eta_1\nu'.$$

The transformations (1) will form a group if, and only if, $T_{\eta'} T_\eta = T_{\eta''}$, where $\eta_1'', \cdots, \eta_n''$ have the values just given, while the relations

$$\alpha'' = \sum_{k=1}^{n} a_k \eta_k'', \quad \beta'' = \sum_{k=1}^{n} b_k \eta_k'', \quad \cdots$$

reduce to identities in η_i and η_i'. Upon replacing α'', β'', $\cdots$, η_k'' by the above values. Comparing the coefficients of $\eta_1\eta_1'$, we find that a_1, b_1, c_1, l_1, m_1, n_1 are zero. For $n = 5$, the remaining conditions are

$$(4) \quad a_3a_2 + a_4b_2 + a_5l_2 = 0, \quad a_4c_2 + a_5m_2 = 0, \quad a_5n_2 = 0,$$

$$(5) \quad c_3a_2 + c_4b_2 + c_5l_2 = 0, \quad c_4c_2 + c_5m_2 = 0, \quad c_5n_2 = 0,$$

$$(6) \quad n_3a_2 + n_4b_2 + n_5l_2 = 0, \quad n_4c_2 + n_5m_2 = 0, \quad n_5n_2 = 0,$$

$$(7) \quad \begin{cases} b_3a_2 + b_4b_2 + b_5l_2 = a_2c_2, \quad b_4c_2 + b_5m_2 = a_3c_2, \\ b_5n_2 = a_4c_2, \quad 0 = a_5c_2, \end{cases}$$

$$(8)\quad \begin{cases} m_3a_k + m_4b_k + m_5l_k = c_2n_k, \quad m_4c_k + m_5m_k = c_5n_k, \\ m_5n_k = c_4n_k, \quad 0 = c_5n_k, \end{cases}$$

$$(9)\quad \begin{cases} l_3a_k + l_4b_k + l_5l_k = a_2m_k + b_2n_k, \quad l_4c_k + l_5m_k = a_5m_k + b_5n_k, \\ l_5n_k = a_4m_k + b_4n_k, \qquad 0 = a_5m_k + b_5n_k, \end{cases}$$

where k takes the values 2, 3, 4, 5.

3. For $n = 3$, the a_k are the only coefficients to be considered, and the preceding conditions reduce to $a_2a_k = 0 (k = 2, 3)$. Hence $\alpha = a_2\eta_2$. Then η_1'', η_2''. η_3'', α'' are symmetrical in η_i and η_i', so that the group is abelian.

4. For $n = 4$, l_k, m_k, n_k do not occur, so that conditions are

$$a_3a_k + a_4b_k = 0, \quad a_4c_k = 0, \quad c_3a_k + c_4b_k = 0, \quad c_4c_k = 0,$$
$$b_3a_k + b_4b_k = a_3c_k, \quad b_4c_k = a_3c_k, \quad 0 = a_4c_k,$$
$$(k = 2, 3, 4).$$

Hence $c_4 = 0$. If either c_2 or c_3 is not zero, the conditions reduce to

$$a_4 = 0, \quad b_4 = a_3 = 0, \quad c_3a_2 = 0, \quad b_3a_2 = a_3c_2.$$

If $a_2 \neq 0$, then $\alpha = a_2\eta_2$, $\beta = b_2\eta_2 + b_3\eta_3$, $\gamma = b_3\eta_3$, and η_j'', α'', β'', γ'' are symmetrical in η_i and η_i'. The group is therefore abelian. If $a_2 = 0$, T_η takes the form (3). The group G is abelian if, and only if, $b_3 = c_2$. If* $b_3 \neq c_2$, the only "ausgezeichnete" infinitesimal transformations are the $e_1Y_1 + e_4Y_4$.

Let next $c_2 = c_3 = 0$, so that the conditions are

$$a_3a_k + a_4b_k = 0, \quad b_3a_k + b_4b_k = 0 \qquad (k = 2, 3, 4).$$

If $a_4 = 0$, then $a_3 = b_4 = 0$, $b_3a_2 = 0$. If also $a_2 = 0$, T_η is of the form (3) with $c_2 = c_3 = 0$. But if $a_2 \neq 0$, then $\alpha = a_2\eta_2$, $\beta = b_2\eta_2$, $\gamma = 0$, so that the group is abelian. Finally, if $a_4 \neq 0$, the conditions reduce to the following:

$$a_3 + b_4 = 0, \quad a_3^2 + a_4b_3 = 0, \quad a_3a_2 + a_4b_2 = 0,$$

whence $\beta = -\alpha a_3/a_4$, $\gamma = 0$, so that T_η is of the form (2). The group G is then not abelian (§1). To bring it to the reduced form, set

* The group is of the type (V′), page 588, Lie-Scheffers, Continuierliche Gruppen.

$$x_3 = a_3\xi_3 + a_4\xi_4, \quad \zeta_3 = a_3\eta_3 + a_4\eta_4.$$

Then T_η becomes

$$\xi_1' = \eta_1\xi_1, \quad \xi_2' = \eta_2\xi_1 + \eta_1\xi_2, \quad x_3' = \zeta_3\xi_1 + \eta_1 x_3,$$

$$\xi_4' = \eta_4\xi_1 - \frac{a_3}{a_4}(a_3\eta_3 + \zeta_3)\xi_2 + \eta_1\xi_4.$$

Its self-conjugate transformations are the following:

$$\xi_1' = \eta_1'\xi_1, \quad \xi_2' = \eta_1'\xi_2, \quad x_3' = \eta_1' x_3, \quad \xi_4' = \eta_1'\xi_1 + \eta_1'\xi_4.$$

The group of transformations (1) *on four variables is either abelian or else is one of the types* (2) *and* (3), *whose self-conjugate transformations form groups of exactly two parameters.*

5. Let next $n = 5$. Then $n_5 = 0$. If $a_5 \neq 0$, the last of the conditions (4) and (7) give $n_k = 0$, $c_k = 0$, and the second condition (4) gives $m_k = 0$. Hence $\gamma = \mu = \nu = 0$. The first condition (4) gives $a_3\alpha + a_4\beta + a_5\lambda = 0$. Set

$$x_3 = a_3\xi_3 + a_4\xi_4 + a_5\xi_5, \quad \zeta_3 = a_3\eta_3 + a_4\eta_4 + a_5\eta_5.$$

Then $x_3' = \zeta_3\xi_1 + \eta_1 x_3$, so that, by applying a transformation on ξ_3, ξ_4, ξ_5 and a transformation on the parameters η_3, η_4, η_5, we obtain a transformation (1) with $\alpha = 0$.

Let $a_5 = 0$, $a_4 \neq 0$. Then $c_4 = 0$, so that $\gamma = 0$, and $a_3\alpha + a_4\beta = 0$. Set

$$x_3 = a_3\xi_3 + a_4\xi_4, \quad \zeta_3 = a_3\eta_3 + a_4\eta_4.$$

Then $x_3' = \zeta_3\xi_1 + \eta_1 x_3$. If $a_5 = a_4 = 0$, then $a_3 a_k = 0$ by (4), so that $a_3 = 0$.

Let $a_5 = a_4 = a_3 = 0$, $c_5 \neq 0$. Then $n_k = 0$ by the third equation (5), so that $\nu = 0$. Also $c_3 a_3\eta_3 + c_4\beta + c_5\lambda = 0$, $c_4\gamma + c_5\mu = 0$ by the first and second equations (5). Set

$$x_4 = c_4\xi_4 + c_5\xi_5, \quad \zeta_4 = c_4\eta_4 + c_5\eta_5.$$

Then $x_4' = \zeta_4\xi_1 - c_3 a_3\eta_3\xi_2 + \eta_1 x_4$. Hence, by applying a transformation on ξ_4, ξ_5 and one on the parameters η_4, η_5, we obtain a transformation (1) with $\gamma = 0$.

Let $a_5 = a_4 = a_3 = 0$, $c_5 = 0$. Then $c_4 = 0$ by (5). If $b_5 \neq 0$, then $n_k = 0$ by the third equation (7), so that $\nu = 0$. By the first and second equations (7),

$$(b_3 - c_3)a_3\eta_3 - a_3 c_3\eta_3 + b_4\beta + b_5\lambda = 0, \quad b_4\gamma + b_5\mu = 0.$$

Hence

$$x_4' = \zeta_4\xi_1 + [(c_3 - b_3)a_3\eta_3 + a_3c_3\eta_3]\xi_2 + \eta_1 x_4,$$
$$x_4 \equiv b_4\xi_4 + b_5\xi_5,\ \zeta_4 \equiv b_4\eta_4 + b_5\eta_5.$$

We may therefore take $b_5 = 0$. Then $b_4b_5 = 0$ by the first equation (7), so that $b_4 = 0$. Then $m_5 = 0$ by the second equation (8). Hence $l_5 = 0$ by the first equation (9). Hence T_7 becomes

$$\xi_1' = \eta_1\xi_1,\quad \xi_2' = \eta_2\xi_1 + \eta_1\xi_2,\quad \xi_3' = \eta_3\xi_1 + a_3\eta_3\xi_2 + \eta_1\xi_3,$$
$$\xi_4' = \eta_4\xi_1 + (b_3\eta_3 + b_4\eta_4)\,\xi_2 + (c_3\eta_3 + c_4\eta_4)\,\xi_3 + \eta_1\xi_4,$$
$$\xi_5' = \eta_5\xi_1 + (l_2\eta_2 + l_3\eta_3 + l_4\eta_4)\,\xi_2 + (m_2\eta_2 + m_3\eta_3 + m_4\eta_4)\,\xi_3$$
$$+ (n_2\eta_2 + n_3\eta_3 + n_4\eta_4)\,\xi_4 + \eta_1\xi_5.$$

Since the group is now in its reduced form, it contains the self-conjugate transformations, in which η_1' and η_5' are arbitrary, $\eta_1' \neq 0$,

$$\xi_i' = \eta_1'\xi_i\,(i = 1, 2, 3, 4),\quad \xi_5' = \eta_5'\xi_1 + \eta_1'\xi_5.$$

6. The conditions (4)–(9) on T_7 in its reduced form are

$$c_3a_3 = 0,\quad n_3a_3 + n_4b_3 = 0,\quad n_4b_3 = 0,\quad n_4c_3 = 0,\quad n_4c_4 = 0,$$
$$b_3a_3 = a_3c_3,\quad m_3a_3 + m_4b_3 = c_3n_3,\quad m_4b_3 = c_3n_3,\quad m_4c_3 = c_3n_3,$$
$$m_4c_4 = c_4n_4,\quad l_3a_3 + l_4b_3 = a_3m_3 + b_3n_3,\quad l_4b_3 = a_3m_3 + b_3n_3,$$
$$0 = a_3m_4 + b_3n_4,\quad l_4c_3 = b_3n_3,\quad l_4c_4 = b_4n_4.$$

If $n_4 \neq 0$, then $b_3 = c_3 = c_4 = 0$, $n_3a_3 + n_4b_3 = 0$. Set

$$x_4 = n_3\xi_3 + n_4\xi_4,\quad \zeta_4 = n_3\eta_3 + n_4\eta_4.$$

Then $x_4' = \zeta_4\xi_1 + \eta_1 x_4$. Hence by introducing x_4 in place of ξ_4, and ζ_4 in place of η_4, T_7 retains its reduced form and has $b_3 = b_4 = c_3 = c_4 = 0$. Then

(10) $$n_3a_3 = 0,\quad m_3a_3 = 0,\quad l_3a_3 = m_3a_3,\quad m_4a_3 = 0,$$

are the only further conditions.

If $a_3 \neq 0$, we obtain the transformation

$$\xi_1' = \eta_1\xi_1,\quad \xi_2' = \eta_2\xi_1 + \eta_1\xi_2,\quad \xi_3' = \eta_3\xi_1 + a_3\eta_3\xi_2 + \eta_1\xi_3,$$
$$\xi_4' = \eta_4\xi_1 + \eta_1\xi_4,$$
$$\xi_5' = \eta_5\xi_1 + (l_2\eta_2 + l_3\eta_3 + l_4\eta_4)\xi_2 + l_3\eta_2\xi_3 + (n_3\eta_2 + n_4\eta_4)\xi_4 + \eta_1\xi_5.$$

It is readily verified that these transformations form a group with the parameters $\eta_1 \cdots, \eta_5$, whatever be the values of $a_2, l_2, l_3, l_4, n_3, n_4$. The expressions for $\eta_i''(i = 1, 2, 3, 4)$ are symmetric in η_i and η_i', but that for η_5'' is symmetric if, and only if, $n_2 = l_4$. In the latter case only, the group is abelian. For $n_2 \neq l_4$, a transformation will belong also to the reciprocal group if, and only if, $\eta_2 = \eta_4 = 0$. Hence the subgroup of self-conjugate transformations has three arbitrary parameters η_1, η_3, η_5.

But, if $a_2 = 0$, the conditions (10) become identities. Hence the transformations

$$T_\eta \text{ with } a_2 = b_2 = b_3 = c_2 = c_3 = 0, \quad n_4 \neq 0,$$

form a group, whatever be the values of l_i, m_i, n_i. It is abelian if, and only if, $m_2 = l_3$, $n_2 = l_4$, $n_3 = m_4$. A self-conjugate transformation must have

$$\eta_2 = \eta_3 = 0, \quad \text{if } m_2 \neq l_3; \quad \eta_2 = \eta_4 = 0, \text{ if } n_2 \neq l_4;$$
$$\eta_3 = \eta_4 = 0, \quad \text{if } n_3 \neq m_4.$$

If G is not abelian, the subgroup of its self-conjugate transformations has two or three arbitrary parameters.

Let next $n_4 = 0$. If $a_2 \neq 0$, the conditions are

$$c_3 = n_3 = m_4 = 0, \quad b_3 = c_2, \quad (l_3 - m_2)a_2 = (n_2 - l_4)b_2,$$
$$m_3 a_2 = c_2 n_2 = l_4 c_2.$$

If also $n_2 = l_4$, so that $l_3 = m_2$, then $\eta_j''(j = 1, \cdots, 5)$ is symmetric in η_i and η_i' and the group is abelian. If $n_2 \neq l_4$, and $l_3 = m_2$, then $c_2 = 0$, $b_2 = 0$, $m_3 = 0$, so that

$$\xi_4' = \eta_4\xi_1 + \eta_1\xi_4, \quad \xi_5' = \eta_5\xi_1 + (l_2\eta_2 + l_3\eta_3 + l_4\eta_4)\xi_2$$
$$+ l_2\eta_2\eta_3 + n_2\eta_2\xi_4 + \eta_1\xi_5,$$

with the restrictions $a_2 \neq 0$, $n_2 \neq l_4$. The self-conjugate transformations have $\eta_2 = \eta_4 = 0$, η_1, η_3, η_5 arbitrary. If $n_2 \neq l_4$, and $l_3 \neq m_2$, then $c_2 = m_3 = 0$, and the self-conjugate transformations have $\eta_2 = \eta_3 = \eta_4 = 0$, η_1 and η_3 arbitrary.

Let next $n_4 = a_2 = 0$. The conditions are

$$m_4 b_2 = c_2 n_3, \quad m_4 b_3 = c_3 n_3, \quad m_4 c_2 = c_3 n_2, \quad (m_4 - n_3)c_3 = 0,$$
$$l_4 b_3 = b_2 n_3, \quad l_4 c_2 = b_3 n_2, \quad l_4 c_3 = b_3 n_3, \quad (l_4 - n_2)b_2 = 0.$$

The transformations form an abelian group if, and only if

$$b_2 = c_2, \quad l_3 = m_2, \quad l_4 = n_2, \quad m_4 = n_3, \quad n_3 c_2 = c_3 n_2, \quad n_3 b_2 = c_2 n_2.$$

The form of the general transformation can be simplified by applying a transformation on ξ_2, ξ_3, and the cogredient transformation on η_2, η_3, and similarly a transformation on ξ_4, ξ_5 and one on η_4, η_5.

7. The argument of Burnside, l. c., §6, page 553, is faulty. It does not show that $\nu = \mu$, but does prove that ν is a multiple of μ. In view of the work of Frobenius and that of Molien, the theorem in question is true.

THE UNIVERSITY OF CHICAGO,
May 12, 1902.

ERRORS IN LEGENDRE'S TABLES OF LINEAR DIVISORS.

BY DR. D. N. LEHMER.

SOME years ago an error in Legendre's Tables of Linear Forms came to my notice. Another was found recently by members of my class, and as this error was left without correction in the later editions I determined to make a careful computation of the whole set. I was surprised to find the list of errors so long. The importance of these tables for many investigations makes it desirable that all these corrections be noted. I have also compared results with the tables in Tshebyshef's Theorie der Congruenzen, Berlin, 1889. Most of the errors in Legendre's work have been carried over uncorrected into these tables.

I. Under the form $t^2 - 29u^2$ the form $116x + 3$ should read $116x + 7$. This error was corrected in the fourth edition (1900), which is a copy of the edition of 1830.

II. Under the form $t^2 - 38u^2$ the form $152x + 129$ should read $152x + 131$. Not corrected in the fourth edition nor in Tshebyshef.

III. Under the form $t^2 - 43u^2$ the form $172x + 147$ should read $172x + 137$. Not corrected in the fourth edition nor in Tshebyshef.

IV. Under $t^2 - 51u^2$ there are two forms $204x + 13$. The second of these should read $204x + 31$. This error is in the fourth edition but not in the first (1797).

V. Under $t^2 - 61u^2$ there are so many errors that I will give the correct list: $244x + 1$, 3, 5, 9, 13, 15, 19, 25, 27, 39, 41, 45, 47, 49, 57, 65, 73, 75, 77, 81, 83, 95, 97, 103, 107, 109, 113, 117, 119, 121, 123, 125, 127, 131, 135, 137, 141, 147, 149, 161, 163, 167, 169, 171, 179, 187, 195, 197, 199, 203, 205, 217, 219, 225, 229, 231, 235, 239, 241, 243. The

numerous errors in this list appear in the first and fourth editions, but are corrected in Tshebyshef.

VI. Under $t^2 - 62u^2$ the form $248x + 103$ should read $248x + 107$. Not corrected in the fourth edition nor in Tshebyshef.

VII. Under $t^2 - 73u^2$ the form $292x + 99$ should read $292x + 69$. This error does not occur in the first edition.

VIII. Under $t^2 - 77u^2$ there are two forms $308x + 53$. The second of the two should be replaced by $308x + 137$. There are also two forms $308x + 255$. The second should be replaced by $308x + 171$. These errors appear in the first and fourth editions, but are corrected in Tshebyshef.

IX. Under $t^2 + 61u^2$ the form $244x + 215$ is omitted in the fourth edition and also in Tshebyshef. This error is not in the first edition.

X. Under the form $t^2 + 77u^2$ there are a number of errors. In the first edition the incorrect forms $308x + 89$, $308x + 149$ and $308x + 257$ appear, and the form $308x + 113$ is repeated. In the fourth edition the corrections $308x + 61$, $308x + 101$, $308x + 153$, $308x + 297$ and $308x + 119$ are made. These corrections are right except the last two which should read $308x + 237$ and $308x + 159$. Tshebyshef is equally unfortunate in his correction of this list. He brings in the incorrect forms $308x + 119$ and $308x + 143$, and omits the correct form $308x + 237$. His list thus contains one too many forms.

XI. Under the form $t^2 + 101u^2$ the forms $404x + 305$, $404x + 313$, $404x + 321$ and $404x + 329$ should be replaced by the forms $404x + 309$, $404x + 317$, $404x + 325$ and $404x + 333$. These errors are in all the tables.

XII. Under the form $t^2 + 91u^2$ the form $182x + 7$ should read $182x + 115$. Occurs in all the tables.

XIII. Under the form $t^2 + 74u^2$ the form $296x + 299$ is used instead of the equivalent simpler form $296x + 3$. This error is noted in the list of errata in the first edition and does not appear elsewhere.

BERKELEY, CALIFORNIA,
April, 1902.

SOME RECENT BOOKS ON MECHANICS.*

A Treatise on Physics. By ANDREW GRAY, LL.D., F.R.S. Vol. 1, *Dynamics and Properties of Matter.* London, J. & A. Churchill, 1901. 8vo., xxiii + 688 pp.

Cours de Mécanique. Par CH. CELLÉRIER. Paris, Gauthier-Villars et Fils, 1892. viii + 617 pp.

NEARLY half a century ago Professors William Thomson of Glasgow and P. G. Tait of Edinburgh planned a Treatise on Natural Philosophy. This work, to be published in three volumes, was to contain a thorough discussion of the phenomena of physical science. The plan seems to have been too great to carry through. The first volume was written and published. Later it was reprinted in two large parts—but alone; and the hope of the scientific world that the other two volumes might be forthcoming must needs be given up forever. Now Professor Andrew Gray, Lord Kelvin's successor in the chair of natural philosophy at Glasgow, publishes the first volume of what purposes to be a three volume Treatise on Physics—a sort of junior Treatise on Natural Philosophy as it seems—which, in the words of the author, shall "serve for those who, beginning at the elements of the subject, wish to have in one book an account of theoretical and experimental physics which may be sufficient for most practical purposes of a scientific and technical education."

How like these sentiments are to those which preface the older work! Yet how the conditions have changed! In the sixties there were in English few if any valuable treatises on mechanics or physics; that of Thomson and Tait stood alone. Now there are numerous expositions of every branch of physical science whether experimental or theoretical. Professor Gray himself has provided us with three volumes on Absolute Measurements in Electricity and Magnetism, one volume on Magnetism and Electricity, and a volume on Bessel's Functions and their Applications with the coöperation of Professor Mathews. But the changes in the number of the expositions of science do not compare with the changes in science itself. During the past thirty or forty years so great advances have been made that for one or two authors to compose something which shall be more than a collection of facts and theories, something which

* Continued from the May number of the BULLETIN.

shall carry the reader through a well planned argument from the rudiments to the outmost confines of science and shall place him in immediate touch with research, seems now an impossible task. Consequently Professor Gray limits the scope of his book in the passage we have cited above. As it is, the scope remains sufficiently large, the expanse sufficiently broad and deep, to tax the powers of even the most gifted.

The first volume of this new treatise on physics bears the title Dynamics and Properties of Matter and is intended to set forth all those general principles and theorems of mechanics which may be useful later in the discussions of acoustics, optics, thermodynamics, electricity and magnetism. Some few subjects however, which have a more special than general application, have been held over for later volumes where they will be developed as needed. Such for example are the dynamical theories of osmosis, diffusion, viscosity and the principle of dimensions or dynamical similarity. With these exceptions the volume omits little that could be desired of kinematics, mechanics, hydrodynamics, elasticity, or capillarity.

The mathematical and physical difficulties encountered in reading the book vary so greatly from chapter to chapter, even from section to section, that unless one looks carefully he may be tempted to think that the author has no definite point of view, has merely collected and edited hastily a mass of facts and theorems bearing on physical science. This however is not the case. The author's intention to make his book useful to the greatest possible number has necessarily led him to include a large variety of subjects treated from an elementary standpoint and suited to students or engineers whose knowledge of theoretical mathematics does not extend far beyond a first course in calculus. To have omitted the difficult portions would have meant giving up the aim of the treatise. Consequently we find intermingled sections of all degrees of difficulty. The most difficult are marked by an asterisk which serves not only to warn off the reader but to inform him that what is there developed will not be made use of in any future unmarked sections.

In this manner Professor Gray has produced a work which cannot fail to be of great use. For heretofore if one wished a knowledge of even the elements of dynamics of a particle, rigid dynamics, hydromechanics, and elasticity he would probably be compelled to look into a half dozen books written by different authors, in different notations, and from different standpoints; if furthermore he wished to

learn the more advanced portions of those subjects he would certainly have to lose much time in unnecessarily scattered reading. Fortunately now teachers at the end of a short course in differential and integral calculus can put into the hands of ambitious students this work of Professor Gray which will serve to keep pace for two or three years with the growing mind and enlarging knowledge of the student. To the teacher the book is also valuable owing to its almost encyclopedic nature. It does not do away with special treatises on the various branches of mathematical physics, but it does contain in a connected manner most of the elementary and intermediate parts of these works.

In a short introduction Professor Gray sets forth some statements and points of view that are so true of the work in hand and so untrue of other works as to be worthy of note. After calling attention to the lamentable fact that the study of mechanics has degenerated too much into the mere solving of more or less artificial and tediously algebraic problems, he states that he purposes to deal so far as possible with only such problems as occur and are easily recognizable in nature. He also lays great stress on the point that cause and effect have no place in physics. The force does not cause the motion nor does the motion cause the force—each accompanies the other. The electromotive force and current in a circuit are likewise accompanying, not antecedent and consequent, actions. He shows, however, that the language of cause and effect is often more clear and useful than any other method of speaking and hence ought not to be given up. He finally states that formal definitions and formal logic are not the sole constituents of mathematical physics. More than accurate logic, more than errorless manipulation of analytic processes is essential. The fundamental ideas must be grasped and thoroughly assimilated. These statements the author offers as an apology for what may seem too much profuseness in explanations of the text. The apology is surely unnecessary. The whole book shows the constant and successful attempt to replace formalism by thought. The result is very refreshing inasmuch as the point of view and the little devices of thought are always those of a practical scientist, not those of an ingenious artificer.

After a short chapter of ten pages on the fundamental units, space and time, Chapter II, Kinematics, opens with a discussion of *steps* or vectors and of those properties of centers of gravity which are immediately derivable by means

of steps. Upon page 22 the use of the scalar $i = \sqrt{-1}$ as a quadrantal versor and the multiplication of steps by a general complex scalar are treated. To what purpose this matter has been introduced here is not apparent, unless the author is preparing for the treatment of alternating currents by analytic methods similar to those employed by Professor Steinmetz, of Schenectady. Why no mention of scalar and vector products of vectors is made here or elsewhere in the book is not easy to see. In 1898 Professor Gray states in the introduction to his Treatise on Magnetism and Electricity, "with regard to the mathematical treatment I have, after consideration, decided not to use the vector analysis but to endeavor to insist as far as possible on the physical meaning of the quantities symbolized. Some brevity of expression is no doubt lost by the process but the discussion is on the other hand within the reach of a greater number of readers." Since 1898 Mr. Oliver Heaviside has continued to publish in vector methods, Professor August Föppl has printed his lectures on mechanics in which he constantly employs vectors, and numerous physicists have inclined more and more toward using some form of vector analysis—especially for teaching the elements of mathematical physics. It would seem that in so extended a treatise as this Professor Gray could well make considerable use of vectors, scalar and vector products, and the operators ∇, div., curl, both for abbreviating the analytic expressions and for insisting on the physical meaning of the operations involved. Can it be that Professor Gray has retreated from his position? He has also almost entirely given up that constant insistence on interpretation which was such a successful and valuable feature of his work of 1898. In this respect we think his style has undergone a decided deterioration since writing the unusually clear Treatise on Magnetism and Electricity.

Subtraction of steps and rate of change of steps leads naturally to velocity, acceleration and the hodograph. The hodograph is applied immediately to discuss the motion of a projectile in vacuo and of a point revolving with uniform velocity in a circle. From uniform circular motion there arises a most elaborate discussion of simple harmonic motion —too elaborate for kinematics itself, but amply justifiable by the frequent appearance and great importance of that type of motion in electromagnetic theories. As a generalization, elliptic motion, which is merely the parallel projection of uniform circular motion, is treated. This too is of importance in future theories. The presentation of uniplanar

and spacial kinematics is thorough and admirably brief except for the introduction of the theory of screws. The mention of this subject is quite unnecessary as practically no further use is made of it.

At the beginning of the third chapter, Dynamics, the author takes up the concepts underlying mechanics. He does this not by giving artificial definitions in a deus ex machina fashion, but by pursuing the same line of thought that we suggested in our recent review of Professor Slate's Dynamics. He states Newton's laws of motion in Latin as Newton stated them, translates them, quotes some of Newton's piercing comments which are generally quite overlooked, and makes further comments of his own. The result is a beautiful simplicity and lucidity. No reader can be confused by this part of Professor Gray's work, nor can he fail to be convinced of its essential rightness. In one other point the author sustains us—in reference to the difference or lack of difference between abstract dynamics and geometry. Let us quote a few words from page 107. "Abstract dynamics is a purely ideal science, geometric in a somewhat extended sense caused by the introduction of certain notions not ordinarily employed in purely geometric processes. So long as we confine ourselves to the ideal as we do in geometry there are about it only difficulties of the same kind as we have in geometrical conceptions and these we do not discuss. The special fundamental difficulties arise when the science is applied to the interpretation of nature"

The remainder of the chapter contains the development of the general principles of dynamics—the motion of the mass center, conservation of momentum and of moment of momentum or rate of description of area, the invariable line and plane, momental ellipsoids, pendulums, parallel forces, and impulses. Numerous examples are worked out as illustrations of the principles evolved and methods employed.

The fourth chapter is given over to the discussion and application of Work and Energy. The early introduction of the work done by gravitational forces affords an opportunity for a few preliminary remarks on gravitational potential. The author does not busy himself with computing the work done by numerous system of forces which differ mainly in the degree of complexity of their analytic definition. Instead of this he takes up such real problems as finding the work done in condensing a nebula, in the impact of a jet of particles, in stretching a contractile film,

in blowing a soap-bubble, and in expanding a gas. Thus the student obtains a firm hold on the principle of work and its general applicability to problems he may have to solve later. An elementary discussion of kinetic energy, conservative and dissipative systems, and of friction in particular close the chapter.

The first few sections of chapter V, General Dynamical Methods, deal with the Lagrangian and Hamiltonian equations of motion and with some general dynamical theorems which follow from them. The author once before, in his Treatise on Magnetism and Electricity, has given a treatment of this same subject. In that work thirty pages are devoted to it; in this fewer than half that number. We cannot but regret, and regret deeply, this change. We regarded the earlier presentation of the subject as on the whole the best and clearest to be found in any language. We wished to see the treatment expanded rather than cut down. Space might better be saved in a half dozen other places. The theory of the ignoration of coördinates, of "controllable and uncontrollable" coördinates and of cyclic systems is of the greatest importance in thermodynamics and electromagnetism. Furthermore, discussions of a general nature, such as are set forth in Professor J. J. Thomson's Applications of Dynamics to Physics and Chemistry, are extremely useful in handling systems concerning the minute mechanical construction of which we know nothing and would best assume nothing This general method of work has not attracted so much attention and consideration as it deserves. References to Professor Thomson's book are all too few and Professor Gray's treatment of that particular phase of dynamical theory in his Magnetism and Electricity was especially welcome. He can have little excuse for omitting the subject here unless he intends to discuss it specially in the later volumes of his work. The complete omission of the study of small oscillations is also difficult to account for. The remainder of chapter V is more elementary and is independent of the foregoing part. The methods of treating dynamics of rigid bodies by Euler's angles and Euler's equations of motion, the motion of gyrostats, and motion relative to the earth are presented.

The next two chapters discuss Statics. In the sixth the analytical conditions of equilibrium of bodies, whether at rest or in steady motion, are derived from the principles of work and energy. In the seventh graphical statics is touched on. The eighth chapter, which treats the equilibrium and motion of a flexible cord or chain, is a welcome

presentation of this interesting and useful branch of dynamics.

Chapters IX and X, each about sixty pages in length, deal with the Statics and Kinetics of Fluids and Hydrostatics. The first of these contains for the most part the theoretical developments and applications of hydrokinetics. The criticisms made upon chapter V apply here with only slightly diminished force. The treatment has been so much condensed and shortened from that in the author's Magnetism and Electricity as to be neither so clear nor so valuable. This shortening is probably due in part at least to the necessity of introducing a mass of more elementary descriptive explanations of different phenomena in the kinetics of fluids. In these explanations the author is quite happy. The chapter on hydrostatics contains few mathematical developments and is confined for the most part to the discussion of such subjects as a student in elementary theoretical physics or in practical laboratory work may need. The treatment includes the theory of buoyancy, hydrometers, barometers, and mercury air-pumps.

Three chapters, XI, XII, XIII, deal with matters immediately connected with the law of universal gravitation. The first, Gravitational Attraction, discusses potential problems, computation of potentials due to certain line, surface, or volume distributions of mass, general theories of potential, the theorems of McLaurin, Charles, and Ivory on the attraction of ellipsoidal bodies, and the method of inversion as applied in potential theories. The author strikes a happy mean between giving too much and giving too little. The second of the three chapters, Astronomical Dynamics, is a welcome presentation of the elements of some theories which it is difficult to find except amid a mass of discouraging detail and small numerical corrections. On the resumption of Gravitational Attraction in the thirteenth chapter the specific attraction due to gravity at the earth's surface is taken up from the standpoint of Cavendish's and other more recent experiments for determining the gravitational constant G.

Chapter XIV on The Tides is as welcome as that on astronomical dynamics; for in it we find a quantity of interesting matter difficult to find elsewhere. After showing that those theories of the tides which are based upon mere changes of the moon's position—the so-called equilibrium theories—are inadequate to explain completely the phenomena observed, the author passes on to the theory of tidal actions considered as due to forced vibrations. As

forced vibrations were not treated in the kinematical or dynamical sections of the book, where they properly belong owing to their fundamental importance not only in tidal theories but in acoustics and electromagnetic vibrations, the author is compelled to make a digression on the subject at this point, after which he proceeds to the discussion of instruments for measuring and predicting tides. At the end of the chapter a few considerations of the changes produced by the tides in the earth-moon system give an opportunity for a passing mention of Professor G. H. Darwin's use of tidal theories in cosmology.

The fifteenth and sixteenth chapters are concerned with Elasticity and Capillarity. The presentation of the theories of elasticity is long, seventy-five pages, and varies greatly in difficulty from advanced mathematical considerations of the general equations of motion of isotropic elastic solids, De St. Venant's theories of torsion, the bending and torsion of thin wires, down through the kinematics of homogeneous strain and the theory of linea elastica to such elementary subjects as vibrations due to torsion in a fiber, flexure of beams, stability of struts, Hooke's law, and impact of elastic balls. The chapter on capillarity is useful owing to the careful arrangement and development of general facts about those simple phenomena of capillary action which are known, vaguely at least, to all students of physics.

The last chapter, the seventeenth, is devoted to the discussion of Measurements and Instruments. It may be interesting to note that Thomson and Tait's Natural Philosophy is concluded in a similar manner. There is nothing remarkable in this chapter which contains merely an exposition of the balance, vernier, spherometer and other simple instruments. In fact we should scarcely have mentioned the chapter at all, had it not given a completeness and finish to the volume and had it not been quite in keeping with the whole. The addition of an extensive index covering ten pages printed in small type greatly enhances the value of the work as a book of reference. But we fail to see the value of adding to a volume which is already becoming bulky a fifteen-page catalogue of the works published by Messrs. J. and A. Churchill—especially inasmuch as that catalogue contains not one work on physics or mathematics.

In conclusion let us state that the author's attempt to write a book from which even the least advanced students may get something, intermediate students a very great deal, and advanced students not a little, seems to have been carried out with more success than we should have thought

possible. We hope the future volumes may be as useful and as well done. We wish success to this junior Treatise on Natural Philosophy. May it attain the completion which was denied to its senior.

Professor Cellérier's Cours de Mécanique is of too old a date to belong properly in this series of reviews. The book, however, came to hand only recently and as it exhibits most clearly the great change which has taken place during the last decade in the point of view adopted in teaching mechanics we may be allowed a few words. Ten years ago when this book appeared mechanics began with statics, continued with dynamics of a particle and rigid dynamics, and concluded with hydromechanics. The amount of space given to statics was great—often greater than that given to any two of the other three. A force was something that tried to produce motion or, if it was fortunate, did produce motion. Everything is now changed. Mechanics begins with dynamics. Statics receives in theoretical works not much more than a passing mention. In fact statics is now unduly slighted where before it was given undue emphasis. Force is the rate of change of momentum—neither the cause of motion nor its result, merely an accompanying phenomenon of motion.

Ten years ago we should have welcomed Professor Cellérier's work and recommended it most heartily for several reasons. First, there is throughout the book a striving against artificiality in problems and points of view. The book is evidently written by one who has a knowledge of physics and engineering in addition to theoretical mechanics. Second, the treatment is well arranged, admirably clear, well adapted to students beginning the subject. Fearing that the student may not be so familiarly acquainted with integral calculus as could be desired, the author evaluates a number of double and triple integrals when considering moments of inertia, and proves Green's theorem. Thirdly, the author introduces the elements of elasticity and hydromechanics into his work. Except in French treatises on mechanics, this was unusual ten years ago. We are happy to say that it is less unusual now, though still not sufficiently customary. The old method of teaching the student a great deal of statics, then a great deal of dynamics of a particle, and leaving rigid dynamics with elasticity and hydromechanics to fit in at the end if there be time must be forsaken. The student gets a better hold on general dynamical principles and a far better preparation

for further work whether in physics or mathematics or theoretical mechanics by beginning with a general course which includes the elements of many branches of mechanics than by beginning with a thorough course in any one branch.

Professor Cellérier's Cours de Mécanique was in many ways an interesting and most valuable book when it appeared. We are inclined even now to recommend it to all who wish to follow the method in general use among engineers—the method of starting mechanics with a discussion of statics.

EDWIN BIDWELL WILSON.

YALE UNIVERSITY,
May 10, 1902.

KIEPERT'S CALCULUS.

Grundriss der Differential- und Integral-Rechnung. Von Dr. LUDWIG KIEPERT. I. Theil: *Differential-Rechnung.* 9te Auflage. II. Theil. *Integral-Rechnung.* 7te Auflage. Hannover, Helwingsche Verlagsbuchhandlung, 1900–1901.

A WORK which has gone through so many editions in a country where so much attention is given to mathematical instruction should have its merits; nor in this case will the reader fail to discover them. These are clearness of statement, carefully drawn illustrations in great number, minute explanations, warnings against probable mistakes on the part of the student. Nevertheless from the very fact that we have a ninth reproduction of an older work there is a certain clinging to tradition which impairs the usefulness of the book.

For example, it gives the student a better idea both of the theory and the range of application, if the differential and integral calculus be treated simultaneously; moreover, he begins at once to know something of integration, a matter of great practical importance. Again, a subject so rich in applications to the various sciences and industrial arts should be presented along with these applications. Thus can the subject be made alive to the student rather than a dead and tiresome exercise in formula grinding. It may be, however, that a problem book, used in conjunction with the text, will supply this deficiency, and of course every good teacher will add problems of his own.

The first volume has an introduction concerning functions, limits, infinitesimals of various orders, continuity, the binomial theorem for positive integral exponents, geo-

metrical progression and the number e. It would seem as if the notion of inverse function, always difficult for a student to grasp, required rather more illustration than is given.

Under the head of infinitesimals the derivative is introduced and some illustrations, involving but slight calculation, given from geometry and mechanics. Further such illustrations might well be given, and some of them might precede the definition and the notation. As a rule. notation ought to be brought in only when necessary to express a previously acquired idea.

Continuity and discontinuity are illustrated graphically and then the same cases that gave the graphs are treated algebraically. The chapter closes with a simple proof that e is no rational number.

The calculus proper opens with the differentiation of the algebraic, the exponential and the circular functions. The double sign for the derivatives of the inverse circular functions might be more fully explained, and in introducing the notation

$$dy = df(x) = f'(x)dx,$$

there should be insistence on the fact that the equality has reference to the ratio unity of the two members, as Δx and Δy vanish to dx and dy.

The hyperbolic functions are given a chapter; but it is not noted how by defining circular and hyperbolic functions as the ratios of areas a complete analogy is established between them. (See BULLETIN, Volume 1, page 155.)

The introduction of derivatives of higher orders leads at once to the Taylor-Maclaurin theorem, to which with applications are devoted sixty pages. For careful step by step development this chapter can hardly be surpassed. The theorem is first given for rational algebraic functions, as, for example, the binomial expansion for a positive integral exponent. It is next shown that in certain particular cases, as when the development leads to a decreasing geometric series, R_n, the remainder after n terms, vanishes. The mean-value theorem is then proved and illustrated so that the student is ready for the usual determination of R_n. The importance of the determination is illustrated by considering the increasing geometric series. The usual applications to expansion of functions are given; yet little use is made of various auxiliary methods such as multiplication and division, integration and differentiation of known expansions. This would, to be sure, anticipate matters to be brought up in the chapter on series. But certainly they

could somewhere be well given, and possibly, in an elementary treatise, would be of more value than the rather difficult discussion of the conditions under which the expansion of a binomial whose two terms are numerically equal is valid. A graphic illustration, such as Klein's, of the way in which a function though converging within certain limits diverges without those limits would be an aid to clearness. (See Lamb's Calculus, page 572.)

There follows a good chapter on series. But the treatment is purely algebraic and one misses such geometric illustrations as those in Professor Osgood's pamphlet. The convergence of series of sines and cosines is touched upon. Here it would have been interesting to have made a statement about series that are convergent while the series gotten by differentiating them term by term are not.

The chapter on maxima and minima is remarkable for the great variety of examples worked out. The case where neither x nor y is an implicit function of the other is reserved for a later chapter.

In the treatment of indeterminate forms the usual proof that when $f(x) = \infty$ and $\varphi(x) = \infty$

$$A \equiv \frac{f(x)}{\varphi(x)} = \frac{f'(x)}{\varphi'(x)}$$

is completed by taking account of the possibility that $A = 0$. There is no suggestion that there are other than the usual indeterminate forms; as for example, $\log_1 1$, $\log_\infty \infty$, $\log_0 0$, $\log_\infty 0$, $\log_0 \infty$. "Indeterminate" meaning "yet to be determined" should be distinguished from real indetermination as dy/dx at the origin on the logarithmic spiral

$$\sqrt{x^2 + y^2} = e^{\tan^{-1}(y/x)}.$$

In the chapter on differentiation of implicit functions help would be afforded by a geometrical illustration of the formula

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}.$$

Under interchange of variables, while the simplification of the expression

$$\frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\dfrac{dx^3}{dt^3}}$$

when $x = t$ or when $y = t$ is noted, it is not remarked that the simplification is secured by supposing x or y merely linear in t. The notions of dependent and independent variation need careful definition.

The next two chapters give the usual geometrical applications to tangents, curvature, evolutes, and so on. In the applications to polar coördinates the conception of r as a revolving y-axis and the perpendicular thereto through the origin as a revolving x-axis would enable the student to easily see just how and why the cartesian formulæ pass over into the polar by merely writing dr and rdt in place of dy and dx.

Algebraic investigations on complex numbers, on the roots of equations including methods of approximation thereto, and on determinants occupy the most of 120 pages. The space given to Graefe's method of approximation, since the method requires special conditions for its successful application and inasmuch too as no criterion is furnished for the error in the calculation, might possibly better be occupied with a demonstration that every equation has a root.

The concluding section of the differential calculus deals with functions of several variables. Only the simpler formulæ are derived and the simpler geometrical applications made. Among these are the theory of envelopes of systems of plane curves and the theory of multiple points of plane curves. For the latter the general formula is derived giving the slopes of all the branches through the point in question. Thus, for a triple point, we have

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}\right)^{(3)} \equiv F_{111} + 3F_{112}\frac{dy}{dx} + 3F_{122}\frac{dy^2}{dx^2} + F_{222}\frac{dy^3}{dx^3} = 0,$$

where the subscripts 1 and 2 denote differentiation with regard to x and y respectively.

The Taylor-Maclaurin theorem is extended to functions of several variables and there is a brief treatment of homogeneous functions.

The concluding chapter of the section contains a fairly complete treatment of maxima and minima for functions of several variables.

It is gratifying to find the student warned against supposing that, if two functions of two or more variables vanish with those variables, the function of the higher degree is zero of a higher order. One would suppose the method of Newton's parallelogram old enough so that even English and American texts should not be vitiated by such a conclusion. (See BULLETIN, volume 4, page 535.)

For n variables the conditions for maxima and minima are stated as follows:

$$u = f(x_1, x_2, \cdots, x_n)$$

is a minimum if the first partial derivatives $f_1(x_1, x_2, \cdots x_n)$, $f_2(x_1, x_2, \ldots x_n)$, $\cdots, f_n(x_1, x_2, \cdots x_n)$ *all vanish, and if*

$$D_1 > 0,\ D_2 > 0,\ D_3 > 0, \cdots,\ D_n > 0.$$

Here

$$D_a = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1a} \\ f_{21} & f_{22} & \cdots & f_{2a} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ f_{a1} & f_{a2} & \cdots & f_{aa} \end{vmatrix}$$

for $a = 1, 2, \cdots, n$

On the other hand, u is a maximum if the first partial derivatives vanish as before, and if the determinants D_a *with even index are all positive while those with odd index are all negative.*

The case of maxima or mimima when some of the first partial derivatives become infinite is not considered. Implicit functions are given a special treatment.

The volume contains tables for the Gudermannian relation and for hyperbolic functions, together with a table of 233 important formulæ—practically all those derived or referred to in the text. It is illustrated by 171 carefully drawn diagrams.

The volume on the integral calculus begins with a somewhat detailed explanation of the nature and geometric meaning of an integral. The usual representation of an integral by an area is given so that the derivative is represented by an ordinate. Since in the differential calculus the derivative has been represented by a slope so that the primitive was an ordinate, there should perhaps be greater emphasis laid upon the constancy of the relation in spite of the change in representation.

A demonstration is given, first graphically and then by analysis, that every continuous function has an integral. It is stated, but not proved, nor is an illustration given, that not every continuous function has a derivative.

The various simpler cases of integration are gone over rather rapidly. The author objects to the term "integration by parts," proposing instead "partial integration." But there seems hardly sufficient reason for changing a usage so well established, and besides there is even a better application for the term proposed by him, viz., the integra-

tion of results gotten by partial successive differentiation. Thus the value of

$$\int\int f(xy)dxdy$$

when the limits for y are variable is gotten by first integrating with regard to y as if x were constant. (See Greenhill's Calculus, page 283.)

The applications to quadrature, cubature, rectification and complanation are enriched by many examples.

After the geometric excursus the author returns to the matter of breaking up a fraction into partial fractions giving the demonstration of that possibility with satisfactory thoroughness. In determining the numerators the rather tedious method of undetermined coefficients is exclusively employed, no mention being made of Hermite's division method. (See Greenhill's Calculus, page 387.) If the reader will compare the labor required by the two methods on such a case as breaking up

$$\frac{1}{(x^2-1)^3}$$

into partial fractions he will have no doubt of the great superiority of the less used of the two.

Integration over discontinuities is briefly but instructively treated. A good example is

$$\int_{-a}^{+b} \frac{dx}{x}$$

This is

$$\lim_{\gamma=0} \int_{-a}^{-\gamma} \frac{dx}{x} + \lim_{\delta=0} \int_{+\delta}^{+b} \frac{dx}{x}$$

$$= \lim_{\gamma=0} ln\left(\frac{\gamma}{a}\right) + \lim_{\delta=0} ln\left(\frac{b}{\delta}\right)$$

$$= ln\left(\frac{b}{a}\right) + \lim \cdot ln \cdot \left(\frac{\gamma}{\delta}\right)_{(\gamma=\delta)} = ln\left(\frac{b}{a}\right)$$

$$= \textit{Cauchy's principal value.}$$

The integration of series term by term is made to lead up to the calculation of the elliptic integrals of the first and second kind. Though it is remarked that these integrals have many important applications, none whatever are made.

A number of determinations of definite integrals are given as well as approximations by Simpson's rule and by Gaussian quadrature, the errors being estimated. The more accurate rule of Weddle is not mentioned. A good chapter on planimeters and integraphs would be interesting in this connection.

The error into which one may sometimes fall on substituting y for a function of x under an integral when dy/dx changes its sign within the limits of integration is clearly pointed out by means of the example

$$\int_1^7 (x^2 - 6x + 13)\,dx = 48.$$

If here we make the substitution $y = x^2 - 6x + 13$ we apparently get

$$\pm \tfrac{1}{2} \int_8^{20} \frac{y\,dy}{\sqrt{y-4}} = \pm \tfrac{8}{3}\sqrt{9},$$

an absurdity. Closer examination shows that the minus sign should be used while x varies from 1 to 3 and y from 8 to 4, while the plus sign is right when x varies from 3 to 7 and y from 4 to 20.

The work concludes with an elementary treatment of differential equations, excellent from the algebraic point of view. The proof that every differential equation has an integral is given with great attention to detail. Even if the student is unable to read this, he will at least get an idea of the extreme care necessary if rigorous demonstration is to be had. Difficulties cannot be safely avoided even in the elements; for easy and plausible proofs are liable to become dangerous poisons to the mind that imbibes them. None but geometrical applications are given and these naturally quite fail us when we come to equations of the nth order.

Like the differential calculus, the integral has at the end a collection of all the formulas used. Twelve pages of these formulas deal entirely with differential equations, and if classified under appropriate headings would furnish a good working key.

There should be a table of gamma functions to go with the formulas for definite integrals, and it would seem that if the elliptic integrals are introduced at all there should be tables of E and F, the complete integrals. An index is needed.

ELLERY W. DAVIS.

UNIVERSITY OF NEBRASKA,
May 24, 1902.

CORRECTION.

In our note on "Conics touching the line infinity at one of the circular points," page 122 of the current volume of the BULLETIN, all four foci of the conic are said to coincide with J. As a matter of fact, however, only two of the foci are at J, the other two coinciding with I. This correction involves several slight changes in the wording of the rest of the paper.

E. V. HUNTINGTON,
J. K. WHITTEMORE.

June 3, 1902.

NOTES.

THE Librarian acknowledges the gift to the AMERICAN MATHEMATICAL SOCIETY of one volume presented by Gauthier-Villars, Paris, and one by the Syndics of the University Press, Cambridge, England.

AT the regular meeting of the London mathematical society held on April 10, 1902, the following papers were read: "On divergent series," by Dr. E. W. HOBSON, the president; "Stress and strain in two-dimensional elastic systems," by Professor A. E. H. LOVE; "Further applications of matrix notation to integration problems," by Dr. H. F. BAKER (by title); "On the convergence of series which represent a potential," by Professor T. J. I. BROMWICH (by title); "On the groups defined for an arbitrary field by the multiplication tables of certain finite groups," by Professor L. E. DICKSON (by title). At the meeting May 8, papers were read by Professor W. BURNSIDE on "Groups in which every two conjugate operations are permutable;" by Mr. A. E. WESTERN on "Fermat's theorem on binary powers;" by Mr. H. S. CARSLAW on "The application of contour integration to the solution of problems in the theory of conduction of heat, and to the development of an arbitrary function in series;" by Dr. G. PRASAD on "The use of Fourier's series in the theory of conduction of heat;" and by Dr. F. S. MACAULAY on "Some formulæ in elimination."

AT the meeting of the Edinburgh mathematical society on May 9th, the following papers were read: "Note on the theorems of Menelaus and Ceva," by Dr. J. S. MACKAY; "Notes on decimal coinage and approximation," by Mr.

BUTTERS; "On some systems of conics associated with the triangle," by Professor R. E. ALLARDICE; "The applicability of the law of extensible minors to determinants of special form, by Dr. THOMAS MUIR.

AT the meeting of the Berlin mathematical society on February 26, papers were read by Dr. A. ADLER on "The theory of drawing instruments," and by Dr. G. WALLENBERG on "The theory of differential equations of the first order." At the meeting on March 19 papers were read by Professor G. HAUCK on "Infinite projection," and by Professor K. HENSEL on "Algebraic numbers and analytic functions."

THE next general meeting of the British mathematical association will be held at King's College, London, on October 2, 1902.

JOHNS HOPKINS UNIVERSITY.—The following mathematical courses are announced for the academic year 1902–1903:—By Professor F. MORLEY: Geometry (advanced course), three hours; Theory of groups, two hours, first semester; Theory of functions (advanced course), two hours, second semester; Mathematical seminary, one hour. By Dr. A. COHEN: Lie's theory of differential equations, two hours; Differential geometry, two hours; Elementary theory of functions, two hours; Differential equations, two hours.—By Professor L. S. HULBURT: Projective geometry, four hours; Analytic geometry of three dimensions.

UNIVERSITY OF PARIS.—The following mathematical courses are announced by the Faculty of Sciences for the academic year, 1902–1903:

First semester.—By Professor G. DARBOUX: General principles of infinitesimal geometry.—By Professor E. GOURSAT: Definite integrals and analytic functions.—By Professor P. APPELL: General laws of equilibrium and motion.—By Professor H. POINCARÉ: Theory of tides.—By Professor J. BOUSSINESQ: Theory of undulatory phenomena.—By Professor G. KOENIGS: Kinematics of solid bodies; graphical statics and reciprocal figures; principles of the theory of elasticity.—By Professor BOUTY: Optics.—By Professor H. PELLAT: Thermodynamics.—By Professor L. RAFFY: Principal mathematical theories introductory to courses in science.—By Professor H. ANDOYER: Mo-

tion of rotation of celestial bodies about their centers of gravity.

Second semester.—By Professor E. PICARD: Partial differential equations from the point of view of mathematical physics.—By Professor E. GOURSAT: Differential equations.—By Professor P. APPELL: General laws of motion of systems; analytical mechanics; hydrostatics; hydrodynamics.—By Professor WOLF: Subjects of programme of certificate in astronomy.—By Professor J. BOUSSINESQ: Theory of undulatory phenomena.—By Professor G. KOENIGS: Principles of the resistance of materials; application of theoretical kinematics to the study of machines.

Conferences will be conducted each semester by Professors L. RAFFY, J HADAMARD and P. PUISEUX on the programmes for certificates in higher geometry, infinitesimal calculus, and rational mechanics, respectively.

THE Holland scientific society of Haarlem proposes as the subject for its prize of 1903 the investigation of Japanese mathematics of the middle of the seventeenth century. The subject is particularly interesting to the Dutch nation, since the unusual development of the science in Japan at that time may possibly have been due to influences emanating from Holland.

THE Société scientifique of Brussels announces for its prize competition for 1902 the following topic: To make a critical study of the works of Simon Stevin on mechanics, comparing them with those of Galileo, Pascal and other scientists of the same period.

THE collected physical papers of the late Professor HENRY A. ROWLAND are soon to be published by Johns Hopkins University. The committee of publication consists of President Ira Remsen and Professors W. H. Welch and J. S. Ames.

GINN & Co. announce as in preparation a new edition, to contain about twice as many pages as the last, of The mathematical theory of the Newtonian potential function by Professor B. O. PEIRCE, of Harvard University.

AN encyclopedia of elementary mathematics, in two volumes, by Professor H. WEBER and Dr. J. WELLSTEIN, is announced as in preparation. The work is intended for teachers, and is designed to lay substantial foundations for the elements rather than to enlarge their boundaries.

DR. J. WELLSTEIN, of Strassburg, has been called to the University of Giessen as associate professor of mathematics.

DR. KARL ZSIGMONDY has been made associate professor of mathematics in the Technical High School of Vienna.

PROFESSOR A. C. DIXON has been made professor of mathematics in Queen's College, Belfast.

DR. A. WIMAN has been called from Lund to a professorship of mathematics in the University of Upsala.

PROFESSOR W. H. METZLER, of Syracuse University, has been made a Fellow of the Royal Society of Edinburgh.

MR. H. L. RIETZ, of Cornell University, has been appointed professor of mathematics at Butler College, Irvington, Ind.

DR. A. B. COBLE, of Johns Hopkins University, has been appointed instructor in mathematics at the University of Missouri.

AT the Massachusetts Institute of Technology, the following changes are announced:—Professor JOHN D. RUNKLE, Walker professor of mathematics and senior member of the faculty, also president of the Institute, 1870 to 1878, has been appointed professor emeritus. Dr. C. N. HASKINS has been appointed instructor in mathematics.

PROFESSOR I. L. FUCHS, since 1884 professor of mathematics in the University of Berlin, and of late editor of *Crelle's Journal*, died on April 26, at the age of sixty-eight years.

THE following catalogues of second-hand mathematical books have been received: Mayer and Müller, Prinz Louis Ferdinandstrasse 2, Berlin, fifteen page catalogue of modern works; A. Hermann, 8 Rue de la Sorbonne, Paris, catalogue No. 75, including over two thousand titles of works from the library of the late JOSEPH BERTRAND.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ALEZAIS (R.). Sur une classe de fonctions hyperfuchsiennes et sur certaines substitutions linéaires qui s'y rapportent. Paris, Gauthier-Villars, 1901. 4to. 200 pp. Fr. 10.00

BARBARIN (P.). La géométrie non-euclidienne. Chartres, Durand (Paris, Naud), 1902. 12mo. 79pp. (*Scientia*, physique mathématique, No. 15.) Cloth. Fr. 2.00

BOY (W.). Ueber die curvatura integra und die Topologie geschlossener Flächen. (Diss.) Göttingen, 1901. 8vo. 52 pp.

DOEHLEMANN (K.). Projektive Geometrie in synthetischer Behandlung. 2te, vermehrte und verbesserte Auflage. Leipzig, Göschen, 1901. 12mo. 176 pp. Cloth. (Sammlung Göschen, No. 72.) M. 0.80

DZIOBEK (O.). Lehrbuch der analytischen Geometrie. Teil II: Analytische Geometrie des Raumes. Braunschweig, Graff, 1902. 8vo. 8 + 314 pp. M. 6.00

FRICKE (R.). See PERRY (J.).

GOURSAT (E.). Cours d'analyse de la Faculté des sciences de Paris Vol. I: Dérivées et différentielles; intégrales définies; développements en serie; applications géometriques. (Un premier fascicule de 368 pages est paru.) Paris, Gauthier-Villars, 1902. 8vo. Fr. 17.00

JUNKER (F.). Höhere Analysis. Teil 2: Integralrechnung. 2te, verbesserte Auflage. Leipzig, Göschen, 1901. 12mo. 208 pp. Cloth. (Sammlung Göschen, No. 88.) M. 0.80

——. Repetitorium und Aufgabensammlung zur Differentialrechnung. Leipzig, Göschen, 1902. 12mo. 119 pp. Cloth. (Sammlung Göschen, No. 146.) M. 0.80

——. Repetitorium und Aufgabensammlung zur Integralrechnung. Leipzig, Göschen, 1902. 12mo. 130 pp. Cloth. (Sammlung Göschen, No. 147.) M. 0.80

MIRIMANOFF (D.). Sur les bases du calcul de généralisation. (Diss.) Genève, 1900. 8vo. 80 pp.

PASCAL (E.). Repertorium der höheren Mathematik. Autorisierte deutsche Ausgabe nach einer neuen Bearbeitung des Originals von A. Schepp. II. Geometrie. Leipzig, Teubner, 1902. 8vo. 9 + 712 pp. Cloth. M. 12.00

PERRY (J.). Höhere Analysis für Ingenieure. Autorisierte deutsche Bearbeitung von R. Fricke und F. Süchting. Leipzig, Teubner, 1902. 8vo. 9 + 423 pp. M. 12.00

RUCHONNET (C.). Exposition géométrique des propriétés générales des courbes. 6e édition augmentée et accompagnée de 7 planches. Paris, Gauthier-Villars, 1901. 216 pp., 7 plates. Fr. 2.50

SCHEPP (A.). See PASCAL (E.).

SCHUR (J.). Ueber eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen. (Diss.) Berlin, 1901. 8vo. 74 pp. M. 2.00

SMITH (P. E.). Elementary calculus: a text-book for the use of students in general science. New York, American Book Co., [1902]. 12mo. 2+89 pp. Cloth. $1.25

SÜCHTING (F.). See PERRY (J.).

TSCHEBYSCHEFF (P. L.). Elemente der Zahlentheorie. (Theorie der Kongruenzen.) Deutsch von H. Schapira. Neue, wohlfeile [Titel-] Ausgabe. Berlin, Mayer & Müller, 1902. 8vo. 17 + 314 + 32 pp. M. 4.00

II. ELEMENTARY MATHEMATICS.

BARDEY (E.). Algebraische Gleichungen nebst den Resultaten und den Methoden zu ihrer Auflösung. 5te Auflage, bearbeitet von F. Pietzker. Leipzig, Teubner, 1902. 8vo. 14 + 420 pp. Cloth. M. 8.00

BARTHELS (K. L.).) Lehrbuch der Stereometrie und Trigonometrie in ausführlicher Darstellung. Wiesbaden, 1902. 8vo. 4 + 120 pp. M. 2.80

BOHNERT (F.). Elementare Trigonometrie. Leipzig, 1902. 8vo. 7 + 183 pp. Cloth. M. 2.40

——. Elementare Stereometrie. Leipzig, Schubert, 1902. 8vo. 7+183 pp. Cloth. (Sammlung mathematischer Lehrbücher, herausgegeben von K. Schubert, Vol. IV.) M. 2.40

BORK (H.). Mathematische Hauptsätze für Gymnasien. Anhang. Nach dem Tode des Verfassers bearbeitet von M. Nath. Leipzig, Dürr, 1902. 8vo. 32 pp. M. 0.40

BOS (H.). Geometría elemental. Paris, Hachette, 1902. 16mo. 287 pp. Fr. 3.00

BRIOT (C.)et VACQUANT (C.). Elementos de geometría aplicada. Paris, Hachette, 1902. 16mo. 236 pp., 4 plates. Fr. 3.50

DAVIDSON (J.). Arithmetic and Algebra. London, Hodder, 1902. 8vo. 248 pp. 2s. 6d.

DIEKMANN (J.). See HEILERMANN (H.).

DIEKMANN (J.). Koppe's Geometrie zum Gebrauche an höhern Unterrichtsanstalten vollständig neu bearbeitet. Teil II: Trigonometrie. 18te Auflage (2te Auflage der neuen Bearbeitung). Mit zahlreichen Uebungen und Aufgaben. Essen, Baedeker, 1902. 8vo. 2 + 137 pp. Cloth. M. 2.00

——. Teil III: Die Stereometrie; der Koordinatenbegriff; die Kegelschnitte. 2te Auflage der neuen Bearbeitung. Mit zahlreichen Uebungen und Aufgaben. Essen, Baedeker, 1902. 8vo. 4 + 145 pp., 1 plate. Cloth. M. 2.20

EDERT (R.) und KRÖGER (M.). Geometrie für Mittelschulen und verwandte Anstalten. Mit besonderer Berücksichtigung der zentralen und axialen Symmetrie und des geometrischen Zeichnens. (In 2 Heften.) Heft I: Vorkursus und Planimetrie, erster Teil. Hannover, Meyer, 1902. 8vo. 7 + 91 pp. Boards. M. 1.00

FORMULARIO di algebra. Torino, Bertero, 1901. 16mo. 28 pp.

FOUR-PLACE logarithmic tables; containing the logarithms of numbers and of the trigonometric functions; arranged for use in the entrance examinations of the Sheffield Scientific School of Yale University. New York, Holt, 1902. 8vo. 2 + 29 pp. Cloth. $0.50

FRANKLAND (W. B.). The story of Euclid. London, Newnes, 1902. 12mo. 176 pp. 1s.

FUSS (K.). Resultate und Andeutungen zur Auflösung der Aufgaben aus der Buchstabenrechnung und Algebra. Für Schulen und zum Selbstunterricht bearbeitet. 5te, vermehrte und verbesserte Auflage. Nürnberg, Korn, 1902. 8vo. 7 + 176 pp. M. 1.60

GIRIODI (V.). L'analisi indeterminata di primo grado applicata alla costruzione di un nuovo crivello di Eratostene ed alla ricerca dei minimi divisori dei numeri composti. Alba, Vertamy, 1901. 4to. 32 pp.

GRONAU (K. T. E.). Das Parallelenproblem oder der Beweis des elften euklidischen Axioms als Lehrsatzes. Hagen i. W., Selbstverlag, 1902. 8vo. 7 + 33 pp. M. 2.00

HALLER VON HALLERSTEIN (F.). Lehrbuch der Elementarmathematik. Nach dem Lehrplane für das königl. preussische Kadetten-Korps bearbeitet von B. Hülsen. Teil III: Pensum der Sekunda. 4te Auflage. Berlin, Nauck, 1902. 8vo. 8 + 227 pp. Cloth. M. 4.20

HEILERMANN (H.) und DIEKMANN (J.). Grundlehren der Trigonometrie und Stereometrie. Teil II: Stereometrie, mit zahlreichen Uebungen und Aufgaben. 3te Auflage. Essen, Baedeker, 1902. 8vo. 3 + 43 pp. M. 0.40

HOCH (J.). Das Wichtigste aus der Geometrie. I. Leitfaden der ebenen Geometrie für Gewerbetreibende und gewerbliche Schulen, mit Rücksicht auf die praktischen Anwendungen im gewerblichen und technischen Leben bearbeitet. Leipzig, Klasing, 1902. 8vo. 8 + 80 pp. (L. Huberti's praktische gewerbliche Bibliothek.) Cloth. M. 2.00

HOUEL (J.). Recueil de formules et de tables numériques. 3e édition. Nouveau tirage. Paris, 1901. 8vo. Fr. 4.00

HÜLSEN (B.). See HALLER VON HALLERSTEIN (F.).

INGRAMI (G.). Nozioni di planimetria ad uso delle classi superiori del ginnasio. Bologna, Cenerelli, 1901. 8vo. 67 pp. Fr. 1.25

KAMBLY (L.). Die Elementarmathematik für den Schulunterricht bearbeitet. Teil 3: Ebene und sphärische Trigonometrie; nebst Uebungsaufgaben. 27ste Stereotypauflage. Breslau, Hirt, 1901. 8vo. 4 + 66 pp., 1 plate. Cloth. M. 1.55

KLAUKE (P.). Lehrbuch der Raumlehre für Lehrerbildungs-Anstalten. Nach den neuesten Lehrplänen bearbeitet. Teil I. Für Präparandenschulen. Düsseldorf, Schwann, 1902. 8vo. 8 + 146 pp. M. 1.50

KRÖGER (M.). See EDERT (R.).

LEMOINE (E.). Géométrographie. Paris, Naud, 1902. 8vo. (*Scientia.*) Fr. 2.00

LIEBER (H.) und LÜHMANN (F. VON). Leitfaden der Elementarmathematik, nach den Bestimmungen der preussischen Lehrpläne vom Jahre 1901 neu bearbeitet von C. Müsebeck. Teil I: Planimetrie. Ausgabe A, für Gymnasien, Real-gymnasien und Oberrealschulen. 16te Auflage. Berlin, Simion, 1902. 8vo. 7 + 155 pp. M. 1.50

——. Teil I: Planimetrie. Ausgabe B, für Realschulen, enthaltend die gesamte Lehraufgabe der Mathematik mit Ausnahme der Arithmetik für Realschulen. 2te Auflage. Berlin, Simion, 1902. 8 + 194 pp. M. 1.90

——. Teil III: Ebene Trigonometrie, Stereometrie, sphärische Trigonometrie, Grundlehren von den Koordinaten und Kegelschnitten. 10te Auflage. Berlin, Simion, 1902. 8vo. 7 + 180 pp. M. 180

LÜHMANN (F. VON). See LIEBER (H.).

MILNE (W. J.). Key to "Academic Algebra." New York, American Book Co., [1902]. 8vo. 397 pp. Cloth. $1.50

MÜSEBECK (C.). See LIEBER (H.).

NATH (M.). See BORK (H.).

PIETZKER (F.). See BARDEY (E.).

SCHUBERT (H.). Beispielsammlung zur Arithmetik und Algebra. 2765 Aufgaben, systematisch geordnet. 2te, unveränderte Auflage. 2ter Abdruck. Leipzig, Göschen, 1901. 12mo. 134 pp. Cloth. (Sammlung Göschen, No. 48.) M. 0.80

SEYFFARTH (W.). Allgemeine Arithmetik und Algebra. Zum Gebrauche an höheren Lehranstalten, insbesondere in den mittleren und oberen Klassen der Lehrerseminare herausgegeben. Dresden, Bleyl & Kaemmerer, 1902. 8vo. 119 pp. M. 1.60

TARTINVILLE (A.). Théorie des équations et des inéquations du premier et du second degré à une inconnue, à l'usage des aspirants aux baccalauréats d'ordre scientifique, des candidats aux écoles du gouvernement et des élèves des écoles normales. 3e édition. Paris, Nony, 1902. 8vo. 230 pp.

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